ETS

THE PRAXIS SERIES

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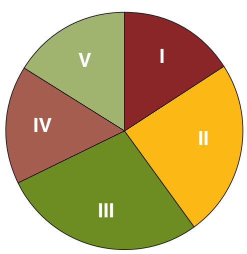
Mathematics: Content Knowledge (0061)

Test at a Glance

Test Name	**Mathematics: Content Knowledge**
Test Code	**0061**
Time	**2 hours**
Number of Questions	**50**
Format	**Multiple-choice questions, graphing calculator required**



	Content Categories	Approximate Number of Questions	Approximate Percentage of Examination
I.	Algebra and Number Theory	8	16%
II.	Measurement	3	6%
	Geometry	5	10%
	Trigonometry	4	8%
III.	Functions	8	16%
	Calculus	6	12%
IV.	Data Analysis and Statistics	5–6	10–12%
	Probability	2–3	4–6%
V.	Matrix Algebra	4–5	8–10%
	Discrete Mathematics	3–4	6–8%

Process Categories	
Mathematical Problem Solving Mathematical Reasoning and Proof Mathematical Connections Mathematical Representation Use of Technology	Distributed Across Content Categories

About This Test

The Praxis Content Knowledge test in Mathematics is designed to assess the mathematical knowledge and competencies necessary for a beginning teacher of secondary school mathematics. Examinees have typically completed a bachelor's program with an emphasis in mathematics or mathematics education.

The examinee will be required to understand and work with mathematical concepts, to reason mathematically, to make conjectures, to see patterns, to justify statements using informal logical arguments, and to construct simple proofs. Additionally, the examinee will be expected to solve problems by integrating knowledge from different areas of mathematics, to use various representations of concepts, to solve problems that have several solution paths, and to develop mathematical models and use them to solve real-world problems. This test may contain some questions that will not count toward your score.

The test is not designed to be aligned with any particular school mathematics curriculum, but it is intended to be consistent with the recommendations of national studies on mathematics education, such as the National Council of Teachers of Mathematics (NCTM) *Principles and Standards for School Mathematics* (2000) and the National Council for Accreditation of Teacher Education (NCATE) *Program Standards for Initial Preparation of Mathematics Teachers* (2003).

Graphing calculators without QWERTY (typewriter) keyboards are required for this test. Some questions will require the use of a calculator. The minimum capabilities required of the calculator are described in the section on graphing calculators. Because many test questions may be solved in more than one way, examinees should decide first how to solve each problem and then decide whether to use a calculator. On the test day, examinees should bring a calculator they are comfortable using.

Selected notations, formulas, and definitions are printed in the test book and are also listed on pages 6–8.

Graphing Calculators

Examinees will be expected to bring to the examination a graphing calculator that can

1. produce the graph of a function within an arbitrary viewing window
2. find the zeros of a function
3. compute the derivative of a function numerically
4. compute definite integrals numerically

These capabilities may be either built into the calculator or programmed into the calculator prior to the examination. Calculator memories need *not* be cleared. Computers, calculators with QWERTY (typewriter) keyboards, and electronic writing pads are NOT allowed.

Unacceptable machines include the following:

Powerbooks and portable/handheld computers

Pocket organizers

Electronic writing pads or pen-input/stylus-driven devices (e.g., Palm, PDA's, Casio Class Pad 300)

Devices with QWERTY keyboards (e.g., TI-92 PLUS, Voyage 200)

Cell-phone calculators

Mathematics Content Descriptions — Basic

Representative descriptions of the topics covered in the basic content categories for the Content Knowledge and the Proofs, Models, and Problems tests follow. Because the assessments were designed to measure the ability to integrate knowledge of mathematics, answering any question may involve more than one competency and may involve competencies from more than one content area.

Algebra and Number Theory

- Demonstrate an understanding of the structure of the natural, integer, rational, real, and complex number systems and the ability to perform the basic operations (+, −, × and ÷) on numbers in these systems
- Compare and contrast properties (e.g., closure, commutativity, associativity, distributivity) of number systems under various operations
- Demonstrate an understanding of the properties of counting numbers (e.g., prime, composite, prime factorization, even, odd, factors, multiples)
- Solve ratio, proportion, percent, and average (including arithmetic mean and weighted average) problems
- Work with algebraic expressions, formulas, and equations; add, subtract, and multiply polynomials; divide polynomials; add, subtract, multiply, and divide algebraic fractions; perform standard algebraic operations involving complex numbers, radicals, and exponents, including fractional and negative exponents
- Solve and graph systems of equations and inequalities, including those involving absolute value
- Interpret algebraic principles geometrically
- Recognize and use algebraic representations of lines, planes, conic sections, and spheres
- Solve problems in two and three dimensions (e.g., distance between two points, the coordinates of the midpoint of a line segment)

Measurement

- Make decisions about units and scales that are appropriate for problem situations involving measurement; use unit analysis
- Analyze precision, accuracy, and approximate error in measurement situations
- Apply informal concepts of successive approximation, upper and lower bounds, and limit in measurement situations

Geometry

- Solve problems using relationships of parts of geometric figures (e.g., medians of triangles, inscribed angles in circles) and among geometric figures (e.g., congruence, similarity) in two and three dimensions
- Describe relationships among sets of special quadrilaterals, such as the square, rectangle, parallelogram, rhombus, and trapezoid
- Solve problems using the properties of triangles, quadrilaterals, polygons, circles, and parallel and perpendicular lines
- Solve problems using the properties of circles, including those involving inscribed angles, central angles, chords, radii, tangents, secants, arcs, and sectors
- Understand and apply the Pythagorean theorem and its converse
- Compute and reason about perimeter, area/surface area, or volume of two- or three-dimensional figures or of regions or solids that are combinations of these figures
- Solve problems involving reflections, rotations, and translations of geometric figures in the plane

Trigonometry

- Define and use the six basic trigonometric relations using degree or radian measure of angles; know their graphs and be able to identify their periods, amplitudes, phase displacements or shifts, and asymptotes
- Apply the law of sines and the law of cosines
- Apply the formulas for the trigonometric functions of $\frac{x}{2},\ 2x,\ x,\ x+y,$ and $x-y;$ prove trigonometric identities
- Solve trigonometric equations and inequalities
- Convert between rectangular and polar coordinate systems

Functions

- Demonstrate understanding of and ability to work with functions in various representations (e.g., graphs, tables, symbolic expressions, and verbal narratives) and to convert flexibly among them
- Find an appropriate family of functions to model particular phenomena (e.g., population growth, cooling, simple harmonic motion)
- Determine properties of a function such as domain, range, intercepts, symmetries, intervals of increase or decrease, discontinuities, and asymptotes
- Use the properties of trigonometric, exponential, logarithmic, polynomial, and rational functions to solve problems
- Determine the composition of two functions; find the inverse of a one-to-one function in simple cases and know why only one-to-one functions have inverses
- Interpret representations of functions of two variables, such as three-dimensional graphs, level curves, and tables

Calculus

- Demonstrate understanding of what it means for a function to have a limit at a point; calculate limits of functions or determine that the limit does not exist; solve problems using the properties of limits
- Understand the derivative of a function as a limit, as the slope of a curve, and as a rate of change (e.g., velocity, acceleration, growth, decay)
- Show that a particular function is continuous; understand the relationship between continuity and differentiability
- Numerically approximate derivatives and integrals
- Use standard differentiation and integration techniques
- Analyze the behavior of a function (e.g., find relative maxima and minima, concavity); solve problems involving related rates; solve applied minima-maxima problems
- Demonstrate understanding of and ability to use the Mean Value Theorem and the Fundamental Theorem of Calculus
- Demonstrate an intuitive understanding of integration as a limiting sum that can be used to compute area, volume, distance, or other accumulation processes
- Determine the limits of sequences and simple infinite series

Data Analysis and Statistics

- Organize data into a suitable form (e.g., construct a histogram and use it in the calculation of probabilities)
- Know and find the appropriate uses of common measures of central tendency (e.g., population mean, sample mean, median, mode) and dispersion (e.g., range, population standard deviation, sample standard deviation, population variance, sample variance)
- Analyze data from specific situations to determine what type of function (e.g., linear, quadratic, exponential) would most likely model that particular phenomenon; use the regression feature of the calculator to determine curve of best fit; interpret the regression coefficients, correlation, and residuals in context
- Understand and apply normal distributions and their characteristics (e.g., mean, standard deviation)
- Understand how sample statistics reflect the values of population parameters and use sampling distributions as the basis for informal inference
- Understand the differences among various kinds of studies and which types of inferences can legitimately be drawn from each
- Know the characteristics of well-designed studies, including the role of randomization in surveys and experiments

Probability

- Understand the concepts of sample space and probability distribution and construct sample spaces and distributions in simple cases
- Understand the concepts of conditional probability and independent events; understand how to compute the probability of a compound event
- Compute and interpret the expected value of random variables in simple cases (e.g., fair coins, expected winnings, expected profit)
- Use simulations to construct empirical probability distributions and to make informal inferences about the theoretical probability distribution

Matrix Algebra

- Understand vectors and matrices as systems that have some of the same properties as the real number system (e.g., identity, inverse, and commutativity under addition and multiplication)
- Scalar multiply, add, subtract, and multiply vectors and matrices; find inverses of matrices
- Use matrix techniques to solve systems of linear equations
- Use determinants to reason about inverses of matrices and solutions to systems of equations
- Understand and represent translations, reflections, rotations, and dilations of objects in the plane by using sketches, coordinates, vectors, and matrices

Discrete Mathematics

- Solve basic problems that involve counting techniques, including the multiplication principle, permutations, and combinations; use counting techniques to understand various situations (e.g., number of ways to order a set of objects, to choose a subcommittee from a committee, to visit n cities)
- Find values of functions defined recursively and understand how recursion can be used to model various phenomena; translate between recursive and closed-form expressions for a function
- Determine whether a binary relation on a set is reflexive, symmetric, or transitive; determine whether a relation is an equivalence relation
- Use finite and infinite arithmetic and geometric sequences and series to model simple phenomena (e.g., compound interest, annuity, growth, decay)
- Understand the relationship between discrete and continuous representations and how they can be used to model various phenomena
- Use difference equations, vertex-edge graphs, trees, and networks to model and solve problems

Mathematical Process Categories

In addition to knowing and understanding the mathematics content explicitly described in the Content Descriptions section, entry-level mathematics teachers must also be able to think mathematically; that is, they must have an understanding of the ways in which mathematical content knowledge is acquired and used. Answering questions on this assessment may involve one or more of the processes described in the Process Categories below, and all of the processes may be applied to any of the content topics.

Mathematical Problem Solving

- Solve problems that arise in mathematics and those involving mathematics in other contexts
- Build new mathematical knowledge through problem solving
- Apply and adapt a variety of appropriate strategies to solve problems

Mathematical Reasoning and Proof

- Select and use various types of reasoning and methods of proof
- Make and investigate mathematical conjectures
- Develop and evaluate mathematical arguments and proofs

Mathematical Connections

- Recognize and use connections among mathematical ideas
- Apply mathematics in context outside of mathematics
- Demonstrate an understanding of how mathematical ideas interconnect and build on one another

Mathematical Representation

- Select, apply, and translate among mathematical representations to solve problems
- Use representations to model and interpret physical, social, and mathematical phenomena
- Create and use representations to organize, record, and communicate mathematical ideas

Use of Technology

- Use technology appropriately as a tool for problem solving
- Use technology as an aid to understanding mathematical ideas

Selected Notations, Formulas, and Definitions (as provided in the test book)

NOTATION

(a, b)	$\{x: a < x < b\}$
$[a, b)$	$\{x: a \le x < b\}$
$(a, b]$	$\{x: a < x \le b\}$
$[a, b]$	$\{x: a \le x \le b\}$
$\gcd(m, n)$	greatest common divisor of two integers m and n
$\operatorname{lcm}(m, n)$	least common multiple of two integers m and n
$[x]$	greatest integer m such that $m \le x$
$m \equiv k \pmod n$	m and k are congruent modulo n (m and k have the same remainder when divided by n, or equivalently, $m - k$ is a multiple of n)
f^{-1}	inverse of an invertible function f (not the same as $\frac{1}{f}$)
$\lim\limits_{x \to a^+} f(x)$	right-hand limit of $f(x)$; limit of $f(x)$ as x approaches a from the right
$\lim\limits_{x \to a^-} f(x)$	left-hand limit of $f(x)$; limit of $f(x)$ as x approaches a from the left
$\varnothing$	the empty set
$x \in S$	x is an element of set S
$S \subset T$	set S is a proper subset of set T
$S \subseteq T$	either set S is a proper subset of set T or $S = T$
$S \cup T$	union of sets S and T
$S \cap T$	intersection of sets S and T

DEFINITIONS

Discrete Mathematics

A relation $\Re$ on a set S is

reflexive if $x \Re x$ for all $x \in S$
symmetric if $x \Re y \Rightarrow y \Re x$ for all $x, y \in S$
transitive if $(x \Re y \text{ and } y \Re z) \Rightarrow x \Re z$ for all $x, y, z \in S$
antisymmetric if $(x \Re y \text{ and } y \Re x) \Rightarrow x = y$ for all $x, y \in S$

An equivalence relation is a reflexive, symmetric, and transitive relation.

FORMULAS

Sum

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

Half-angle (sign depends on the quadrant of $\frac{\theta}{2}$)

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

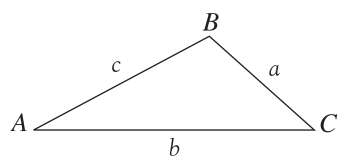
$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

Range of Inverse Trigonometric Functions

$\sin^{-1} x$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[0, \pi]$
$\tan^{-1} x$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



Law of Cosines

$$c^2 = a^2 + b^2 - 2ab(\cos C)$$

DeMoivre's Theorem

$$(\cos\theta + i\sin\theta)^k = \cos(k\theta) + i\sin(k\theta)$$

Coordinate Transformation

Rectangular (x, y) to polar (r, θ): $r^2 = x^2 + y^2$; $\tan\theta = \frac{y}{x}$ if $x \neq 0$

Polar (r, θ) to rectangular (x, y): $x = r\cos\theta$; $y = r\sin\theta$

Distance from point (x_1, y_1) to line $Ax + By + C = 0$

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

Volume

Sphere with radius r: $V = \frac{4}{3}\pi r^3$

Right circular cone with height h and base of radius r: $V = \frac{1}{3}\pi r^2 h$

Right circular cylinder with height h and base of radius r: $V = \pi r^2 h$

Pyramid with height h and base of area B: $V = \frac{1}{3}Bh$

Right prism with height h and base of area B: $V = Bh$

Surface Area

Sphere with radius r: $A = 4\pi r^2$

Right circular cone with radius r and slant height s: $A = \pi rs + \pi r^2$

Differentiation

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

$$(f(g(x)))' = f'(g(x))g'(x)$$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \text{ if } g(x) \neq 0$$

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Sample Test Questions

The sample questions that follow illustrate the kinds of questions in the test. They are not, however, representative of the entire scope of the test in either content or difficulty. Answers with explanations follow the questions.

Directions: Each of the questions or incomplete statements below is followed by four suggested answers or completions. Select the one that is best in each case.

Algebra and Number Theory

1. Jerry is 50 inches tall and is growing at the rate of $\frac{1}{24}$ inch per month. Adam is 47 inches tall and is growing at the rate of $\frac{1}{8}$ inch per month. If they each continue to grow at these rates for the next four years, in how many months will they be the same height?

   (A) 24
   (B) 30
   (C) 36
   (D) 42

2. What is the units digit of 33^{408} ?

   (A) 1
   (B) 3
   (C) 7
   (D) 9

3. If x and y are even numbers and $z = 2x^2 + 4y^2$, then the greatest even number that must be a divisor of z is

   (A) 2
   (B) 4
   (C) 8
   (D) 16

4. A taxicab driver charges a fare of $2.00 for the first quarter mile or less and $0.75 for each quarter mile after that. If f represents the fare, in dollars, which of the following equations models the fare, in dollars, for a ride m miles long, where m is a positive integer?

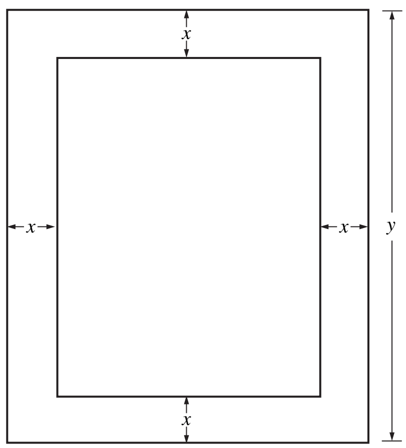
   (A) $f = 2.00 + 0.75(m - 1)$
   (B) $f = 2.00 + 0.75\left(\frac{m}{4} - 1\right)$
   (C) $f = 2.00 + 0.75(4m - 1)$
   (D) $f = 2.00 + 0.75(4(m - 1))$

5. For which of the following values of k does the equation $x^4 - 4x^2 + x + k = 0$ have four distinct real roots?

   I. -2
   II. 1
   III. 3

   (A) II only
   (B) III only
   (C) II and III only
   (D) I, II, and III

Measurement



6. The inside of a rectangular picture frame measures 36 inches long and 24 inches wide. The width of the frame is x inches, as shown in the figure above. When hung, the frame and its contents cover $1{,}408$ square inches of wall space. What is the length, y, of the frame, in inches?

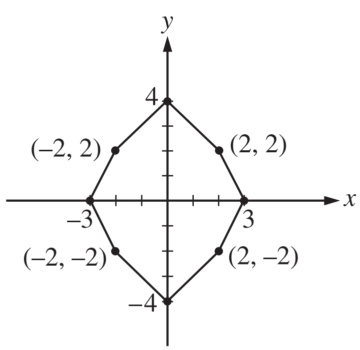
   (A) 44

   (B) 40

   (C) 38

   (D) 34

Geometry



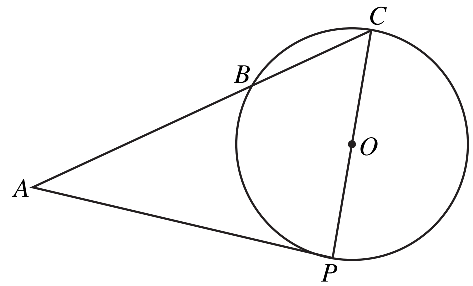
7. For how many angles θ, where $0 < \theta \le 2\pi$, will rotation about the origin by angle θ map the octagon in the figure above onto itself?

   (A) One

   (B) Two

   (C) Four

   (D) Eight



8. In the circle above with center O and radius 2, AP has length 3 and is tangent to the circle at P. If CP is a diameter of the circle, what is the length of BC ?

   (A) 1.25

   (B) 2

   (C) 3.2

   (D) 5

Trigonometry

9. If $y = 5\sin x - 6$, what is the maximum value of y?

   (A) -6

   (B) -1

   (C) 1

   (D) 5

10. In $\triangle ABC$ (not shown), the length of side AB is 12, the length of side BC is 9, and the measure of angle BAC is $30°$. What is the length of side AC?

    (A) 17.10

    (B) 4.73

    (C) 3.68

    (D) It cannot be determined from the information given.

11. In the xy-plane, an acute angle with vertex at the origin is formed by the positive x-axis and the line with equation $y = 3x$. What is the slope of the line that contains the bisector of this angle?

(A) 3

(B) $\frac{3}{2}$

(C) $\frac{\sqrt{10} + 1}{3}$

(D) $\frac{\sqrt{10} - 1}{3}$

Functions

12. At how many points in the xy-plane do the graphs of $y = 4x^5 - 3x^2 - 1$ and $y = -0.4 - 0.11x$ intersect?

(A) One

(B) Two

(C) Three

(D) Four

13. If

(i) the graph of the function $f(x)$ is the line with slope 2 and y-intercept 1

and

(ii) the graph of the function $g(x)$ is the line with slope -2 and y-intercept -1,

which of the following is an algebraic representation of the function $y = f(g(x))$?

(A) $y = 0$

(B) $y = -4x - 3$

(C) $y = -4x - 1$

(D) $y = -(2x + 1)^2$

$$P(t) = 250 \cdot (3.04)^{\frac{t}{1.98}}$$

14. At the beginning of 1990, the population of rabbits in a wooded area was 250. The function above was used to model the approximate population, P, of rabbits in the area t years after January $1, 1990$. According to this model, which of the following best describes how the rabbit population changed in the area?

(A) The rabbit population doubled every 4 months.

(B) The rabbit population tripled every 6 months.

(C) The rabbit population doubled every 36 months.

(D) The rabbit population tripled every 24 months.

15. If $f(x) = 3x^2$, what are all real values of a and b for which the graph of $g(x) = ax^2 + b$ is below the graph of $f(x)$ for all values of x?

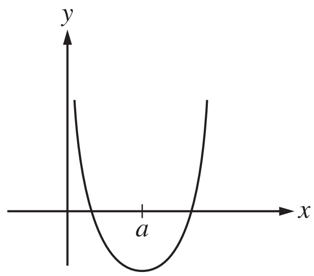
(A) $a \geq 3$ and b is positive.

(B) $a \leq 3$ and b is negative.

(C) a is negative and b is positive.

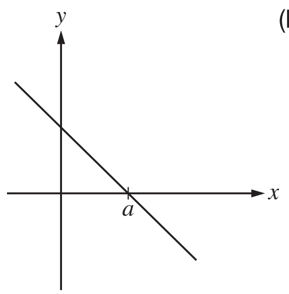
(D) a is any real number and b is negative.

Calculus

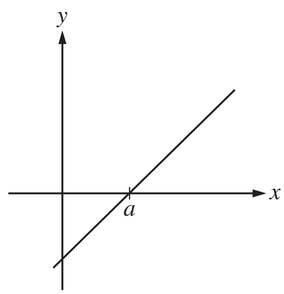


16. The figure above is a graph of a differentiable function f. Which of the following could be the graph of the first derivative of this function?

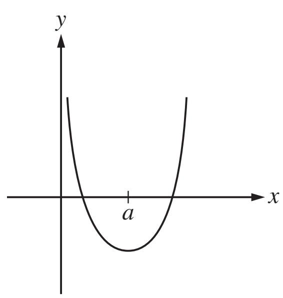
(A)



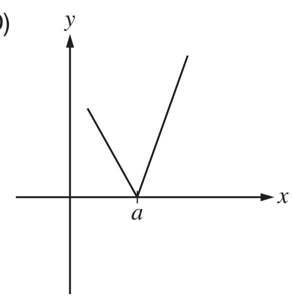
(B)



(C)



(D)



17. If $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = 0$, what can be concluded about the value of $\lim_{x \to c} \frac{f(x)}{g(x)}$?

(A) The value is not finite.

(B) The value is 0.

(C) The value is 1.

(D) The value cannot be determined from the information given.

18. In a certain chemical reaction, the number of grams, N, of a substance produced t hours after the reaction begins is given by $N(t) = 16t - 4t^2$, where $0 < t < 2$. At what rate, in grams per hour, is the substance being produced 30 minutes after the reaction begins?

(A) 7

(B) 12

(C) 16

(D) 20

Data Analysis and Statistics

19. The measures of the hand spans of ninth-grade students at Tyler High School are approximately normally distributed, with a mean of 7 inches and a standard deviation of 1 inch. Of the following groups of measurements of hand span, which is expected to contain the largest number of ninth graders?

(A) Less than 6 inches

(B) Greater than 7 inches

(C) Between 6 and 8 inches

(D) Between 5 and 7 inches

Stem	Leaf						
9	1	3	4	5	7		
8	2	2	5	6	6	8	9
7	0	2	4	5	8	8	
6	1	3	7	9			

20. The stem plot above shows the course grades that each of 22 students received in a history course. The course grade is represented by using the tens digit of each grade as a stem and the corresponding units digit as a leaf. For example, the stem 9 and the leaf 1 in the first row of the table represent a grade of 91. What was the median course grade of the 22 students?

(A) 78

(B) 80

(C) 80.7

(D) 82

Probability

21. A two-sided coin is unfairly weighted so that when it is tossed, the probability that heads will result is twice the probability that tails will result. If the coin is to be tossed 3 separate times, what is the probability that tails will result on exactly 2 of the tosses?

(A) $\frac{2}{9}$

(B) $\frac{3}{8}$

(C) $\frac{4}{9}$

(D) $\frac{2}{3}$

Matrix Algebra

22. The orthogonal projection of 3-space onto the xy-plane takes the point (x, y, z) onto the point $(x, y, 0)$. This transformation can be represented by the matrix equation $M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$, where M is which of the following matrices?

(A) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(B) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(C) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(D) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

23. For what value of x is the matrix $\begin{pmatrix} 1 & 4 \\ x & 6 \end{pmatrix}$ NOT invertible?

(A) $-\frac{3}{2}$

(B) 0

(C) $\frac{3}{2}$

(D) 2

Discrete Mathematics

24. Given the recursive function defined by

$f(1) = -3,$

$f(n) = f(n-1) - 6 \text{ for } n \geq 2,$

what is the value of $f(4)$?

(A) -2

(B) -9

(C) -10

(D) -21

25. For lines in the plane, the relation "is perpendicular to" is

(A) reflexive but not transitive

(B) symmetric but not transitive

(C) transitive but not symmetric

(D) both symmetric and transitive

Answers

1. The heights in this question can be expressed as two linear equations. Jerry's height in inches, J, can be expressed as $J = 50 + \frac{1}{24}m$, where m is the number of months from now. Adam's height in inches, A, can be expressed as $A = 47 + \frac{1}{8}m$. The question asks, "in how many months will they be the same height?" This is the same as asking, "for what value of m will $J = A$?" The solution can be found by solving $50 + \frac{1}{24}m = 47 + \frac{1}{8}m$ for m.

$$50 + \frac{1}{24}m = 47 + \frac{1}{8}m$$
$$50 - 47 = \left(\frac{1}{8} - \frac{1}{24}\right)m$$
$$3 = \left(\frac{3}{24} - \frac{1}{24}\right)m$$
$$3 = \frac{1}{12}m$$
$$m = 36$$

So the correct answer is (C), 36 months.

2. To find the units digit of 33^{408}, it is helpful to find the first few integer powers of 33 and look for a pattern. For example,

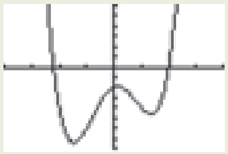
$$33^1 = 33$$
$$33^2 = 1089$$
$$33^3 = 35{,}937$$
$$33^4 = 1{,}185{,}921$$
$$33^5 = 39{,}135{,}393$$

You can see that the pattern in the units digits is $3, 9, 7, 1, 3, \ldots$ and that it will continue to repeat with every four integers of the exponent. Dividing 408 by 4 yields 102 with no remainder. So the units digit of 33^{408} will be the same as the units digit of 33^4, which is 1. So the correct answer is (A).

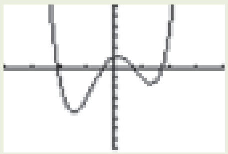
3. Since 2 is a divisor of both $2x^2$ and $4y^2$, it follows that 2 is a divisor of z. To find out if there is a greater even number that must be a divisor of z, you need to consider the additional information given, which is that x and y are both even numbers. Since x and y are even numbers, they can be expressed as $x = 2m$ and $y = 2n$, respectively, where m and n can be either odd or even integers. Substituting these values for x and y into the expression for z yields $z = 2(2m)^2 + 4(2n)^2$. It follows then that $z = 8m^2 + 16n^2$ and that 8 is a divisor of z. The number 16 would also be a divisor of z if m is even, but not if m is odd. Since m and n can be either even or odd and the question asks for the largest even number that must be a divisor of z, the correct answer is 8, answer choice (C).

4. This question asks you to determine which of the four equations given as choices models the fare for a taxi ride of m miles, where m is a positive integer. The question states that the fare is $\$2.00$ for the first quarter mile or less and $\$0.75$ for each quarter mile after that. You will notice by examining the answer choices that all of the choices include a constant term of 2.00 (for the $\$2.00$ for the first quarter mile). Thus, the task is to model the fare for the remaining distance beyond the first quarter mile. Since the question states that $\$0.75$ is charged for each quarter mile after the first, you must determine how many quarter miles the trip is. Since the trip is given as m miles (where m is an integer), the number of quarter miles in the trip would be $4m$. The charge for the first quarter mile is $\$2.00$, so that would leave $4m - 1$ quarter miles to be charged at a rate of $\$0.75$ each. The total fare for the trip would thus be modeled by the equation $f = 2.00 + 0.75(4m - 1)$. By comparing this with the choices given, you will see that the correct answer is (C).

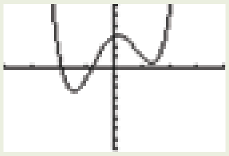
5. You may recall from your study of solutions to polynomial equations that a fourth-degree polynomial has at most four distinct real roots and that the roots of the equation are the x-intercepts of the graph of the equation. One way to determine for which of the three given values of k the equation will have four distinct real roots is to graph the equations using your calculator.



I. $x^4 - 4x^2 + x - 2 = 0$



II. $x^4 - 4x^2 + x + 1 = 0$



III. $x^4 - 4x^2 + x + 3 = 0$

Using an appropriate viewing window to see the behavior of the graphs for the three values of k clearly, you can see that the values of k given in I and III each result in the equation having only two distinct real roots. The value of k given in II results in the equation having four distinct real roots. So the correct answer is II only, answer choice (A).

6. This question requires you to use your knowledge of the area of a rectangle in order to find the outer length, y, of the picture frame described. You should recall that the area of a rectangle can be found by multiplying the length of the rectangle by the width of the rectangle. The inside dimensions of the frame are given as 36 inches long and 24 inches wide. The width of the frame is given as x inches, so that the outside dimensions of the frame would be $36 + 2x$ inches long and $24 + 2x$ inches wide. The area of the rectangle with the outside dimensions of the frame is given as $1{,}408$ square inches. This area can then be represented as $(36 + 2x)(24 + 2x) = 1{,}408$. Solving this for x yields

$$(36 + 2x)(24 + 2x) = 1{,}408$$
$$864 + 48x + 72x + 4x^2 = 1{,}408$$
$$4x^2 + 120x - 544 = 0$$
$$x^2 + 30x - 136 = 0$$
$$(x + 34)(x - 4) = 0$$
$$x = -34 \text{ or } x = 4$$

Only $x = 4$ makes sense in the context of this question, so the width of the frame is 4 inches and therefore the outer length, y, of the frame is $36 + 2x = 36 + 2(4) = 44$. The correct answer is (A), 44 inches.

7. The question asks you to consider rotation about the origin of the octagon in the figure and to determine for how many angles θ, where $0 < \theta \leq 2\pi$, would rotation of the octagon result in the octagon being mapped onto itself. One way to begin is to consider a single point on the octagon, such as the point $(0, 4)$, at the "top" of the octagon in the figure. This point is 4 units from the origin, so any rotation that maps the octagon onto itself would need to map this point onto a point that is also 4 units from the origin. The only other point on the octagon that is 4 units from the origin is the point $(0, -4)$. A rotation of angle $\theta = \pi$ would map the point $(0, 4)$ onto the point $(0, -4)$. You can see that the octagon is symmetric about both the x- and y-axes, so a rotation of angle $\theta = \pi$ would map all of the points of the octagon onto corresponding points of the octagon. Likewise, a rotation of angle $\theta = 2\pi$ would map the point $(0, 4)$ onto itself (and map all other points of the octagon onto themselves). No other values of θ such that $0 < \theta \leq 2\pi$ would map the octagon onto itself. Therefore the correct answer is two, choice (B).

8. To determine the length of BC, it would be helpful to first label the figure with the information given. Since the circle has radius 2, then both OC and OP have length 2 and CP has length 4. AP is tangent to the circle at P, so angle APC is a right angle. The length of AP is given as 3. This means that triangle ACP is a 3-4-5 right triangle and AC has length 5. You should also notice that since CP is a diameter of the circle, angle CBP is also a right angle. Angle BCP is in both triangle ACP and triangle PCB, and therefore the two triangles are similar. You can then find the length of BC by setting up a proportion between the corresponding parts of the similar triangles as follows:

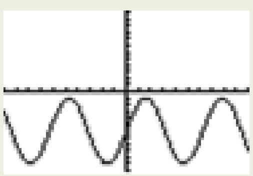
$$\frac{CP}{AC} = \frac{BC}{PC}$$

$$\frac{4}{5} = \frac{BC}{4}$$

$$BC = \frac{16}{5} = 3.2$$

The correct answer, 3.2, is answer choice (C).

9. There are two ways to answer this question. You should be able to use either method. The first solution is based on reasoning about the function $f(x) = \sin x$. First you need to recall that the maximum value of $\sin x$ is 1, and therefore the maximum value of $5\sin x$ is 5. The maximum value of $y = 5\sin x - 6$ is then $5 - 6 = -1$. Alternatively, you could graph the function $y = 5\sin x - 6$ and find the maximum value of y from the graph.



The maximum value is -1, and the correct answer is (B).

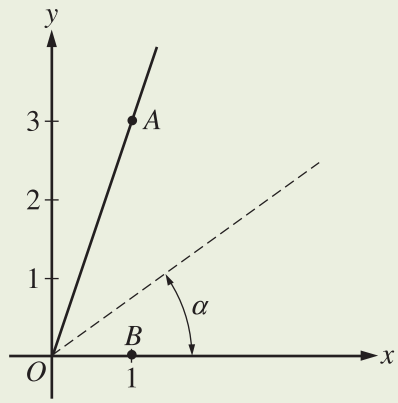
10. In this question, you are given the length of two sides of a triangle and the measure of the angle opposite one of those two sides of the triangle. You are asked to find the length of the third side of the triangle. You should recall that the law of sines relates the lengths of two sides of a triangle and the sines of the angles opposite the sides. (The law of sines is included in the Notation, Definitions, and Formulas pages that are included in this document and at the beginning of each of the Content Knowledge tests.) Using the law of sines yields

$$\frac{\sin(\angle BAC)}{\sin(\angle BCA)} = \frac{BC}{BA} \quad \text{and} \quad \frac{\sin 30°}{\sin(\angle BCA)} = \frac{9}{12}.$$

Therefore, $\sin(\angle BCA) = \frac{4}{3}\sin 30° = \frac{2}{3}$. You should recall that this is an example of the ambiguous case of the law of sines—that since the value of the sine is between 0 and 1, there are two angles between 0 and 180 degrees, one acute and one obtuse, associated with this sine and therefore there are two possible triangles with the given sides and angle measure. The correct answer then is (D), "It cannot be determined from the information given."

The two values of the measure of $\angle BCA$ are approximately $41.8°$ and $138.2°$. Using either the law of sines again (with $\angle BAC$ and $\angle ABC$, or with $\angle BCA$ and $\angle ABC$) or the law of cosines, you can determine that the length of side AC is either approximately 3.68 or 17.10. Since the length of side AC cannot be uniquely determined, the correct answer is (D).

11. To answer this question, it might be helpful to first draw a figure such as the one shown below.



Consider the triangle OAB, where O is the origin, A is the point $(1, 3)$, and B is the point $(1, 0)$. Point A lies on the line $y = 3x$. The acute angle described in the question is the angle AOB. The question asks you to find the slope of the line that contains the angle bisector of angle AOB. Let α be the angle between the x-axis and the angle bisector of angle AOB. Then the slope of the line that contains the angle bisector of angle AOB will be equal to tan α. You can use the half-angle formulas in the Notations, Definitions, and Formulas pages that are included in this document and at the beginning of the Mathematics: Content Knowledge tests to find $\tan \alpha$ in terms of the sine and cosine of angle AOB. From your figure, you can see that $OB = 1, AB = 3,$ and $OA = \sqrt{10}$.

$$\tan \alpha = \frac{\sin\left(\frac{\angle AOB}{2}\right)}{\cos\left(\frac{\angle AOB}{2}\right)} = \frac{\sqrt{\frac{1-\cos(\angle AOB)}{2}}}{\sqrt{\frac{1+\cos(\angle AOB)}{2}}}$$

Simplifying yields

$$\frac{\sqrt{\frac{1-\cos(\angle AOB)}{2}}}{\sqrt{\frac{1+\cos(\angle AOB)}{2}}} \cdot \frac{\sqrt{\frac{1+\cos(\angle AOB)}{2}}}{\sqrt{\frac{1+\cos(\angle AOB)}{2}}} = \frac{\sqrt{\frac{1-\cos^2(\angle AOB)}{2}}}{\frac{1+\cos(\angle AOB)}{2}}$$

$$= \frac{\sin(\angle AOB)}{1+\cos(\angle AOB)}$$

$$= \frac{\frac{3}{\sqrt{10}}}{1+\frac{1}{\sqrt{10}}}$$

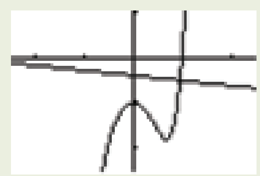
$$= \frac{3}{\sqrt{10}+1}$$

$$= \frac{3\left(\sqrt{10}-1\right)}{10-1}$$

$$= \frac{\sqrt{10}-1}{3}$$

So the correct answer is (D).

12. To answer this question, you should graph the equations on your calculator using an appropriate viewing window and then see how many points of intersection are shown. The figure below shows one view of the intersections of the two graphs.



You should also convince yourself that there are no additional points of intersection that are not visible in this viewing window. One way to do that is to verify that $y = 4x^5 - 3x^2 - 1$ has only two relative extrema, both of which are shown. (Find where $y' = 0$.) Only one point of intersection is shown in the figure above, so the correct answer choice is (A).

13. This question asks you to find an algebraic representation of the composition of the functions $f(x)$ and $g(x)$. First you should write algebraic representations of the individual functions. You are given the slopes and y-intercepts of the lines that are the graphs of $f(x)$ and $g(x)$. Using the slope-intercept form of the equation of a line ($y = mx + b$, where m is the slope and b is the y-intercept) and the information given in parts (i) and (ii) yields the following functions, which have the graphs described in the question: $f(x) = 2x + 1$ and $g(x) = -2x - 1$. These functions imply that $f(g(x)) = 2(-2x - 1) + 1 = -4x - 1$. So $y = -4x - 1$, and (C) is the correct answer.

14. In this question, a model is given for the growth of the rabbit population as a function of time, t, in years. The question asks for a verbal description of the change in the rabbit population, based on the function given. You should recall the meaning of the base (growth factor) and the exponent in an exponential growth model. You should note that the function given $P(t) = 250 \cdot (3.04)^{\frac{t}{1.98}} \approx 250 \cdot 3^{\frac{t}{2}}$. You can observe from this approximation (with base 3, and exponent $\frac{t}{2}$) that the population tripled every two years. Thus the correct answer would be (D), "The rabbit population tripled every 24 months."

15. This question is asking about your understanding of how changing the values of the coefficient a and y-intercept b in a quadratic function $f(x) = ax^2 + b$ affects the graph of the function. You should recall that for $a > 0$, as a decreases, the width of the parabola that is the graph of $y = ax^2$ increases, and for $a < 0$ the graph opens downward. You should also recall that as the value of b decreases, the vertex of the graph of $y = ax^2 + b$ moves in a negative direction along the y-axis. So for the graph of $g(x) = ax^2 + b$ to be below the graph of $f(x) = 3x^2$ for all values of x, a must be less than or equal to 3 and b must be negative (the vertex will be below the vertex of $f(x)$, which is at the origin). The correct answer, therefore, is (B).

16. This question asks you to determine the possible shape of the graph of the first derivative of a differentiable function from the shape of the graph of the function. You should recall that the first derivative of the function at a point is equal to the slope of the graph of the function at that point. By inspection, you will see that, starting near $x = 0$, the slope of the graph of $f(x)$ is negative and becomes less negative as x approaches a and that the slope is 0 at $x = a$ (at the minimum value of f) and then becomes increasingly positive as x increases. Only answer choice (B) is consistent with this behavior. Therefore, (B) is the correct answer.

Alternatively, you may recognize that the graph of the function f appears to be a parabola, which implies that f is a quadratic function. The graph of f suggests that f could be of the form $f(x) = c(x - a)^2 - b$. The first derivative then would be $f'(x) = 2c(x - a)$. The graph of f' would be a line with an x-intercept of a. Since the graph of f is a parabola that opens up, the value of c must be positive (i.e., $f''(x) = 2c > 0$), so the slope of the line that is the graph of f' must also be positive. The only answer choice that is a line with positive slope and x-intercept of a is (B). Therefore, (B) is the correct answer.

17. In a problem such as this, which contains the answer choice "It cannot be determined from the information given," you should be careful to base your answer on correct reasoning. If you conclude that the value can be determined, you should base your conclusion on known mathematical facts or principles; if, however, you conclude that the value cannot be determined, you should support your conclusion by producing two different possible values for the limit.

You should recall that the quotient property of limits states that if $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, and if $M \neq 0$, then $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$. However, this property cannot be used to determine $\lim_{x \to c} \frac{f(x)}{g(x)}$ for the problem at hand since the value of $\lim_{x \to c} g(x)$ is 0 and the quotient property is inconclusive in this case. In fact, for this problem,

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{0}{0}.$$ Note that the expression $\frac{0}{0}$ does not represent a real number; in particular, it is not equal to either 0 or 1. Thus, the value of $\lim_{x \to c} \frac{f(x)}{g(x)}$ cannot be determined by using the basic properties of limits. As a result, you should suspect that, in fact, $\lim_{x \to c} \frac{f(x)}{g(x)}$ cannot be determined and verify your hunch by producing examples to show that the value of the limit depends on the particular functions f and g.

In the remaining discussion, it will be assumed that $c = 0$. (It is always possible to apply a translation of c units to the two functions.)

You should be aware that, although both f and g have the limit 0 as $x \to 0$, one function might be approaching 0 more quickly than the other, which would affect the value of the limit of the quotient. Thus, if one of the functions is x and the other x^2, then the quotient is either x or $\frac{1}{x}$, and so the limit of the quotient is either 0 or nonexistent, respectively. The value of the limit can, in fact, be any nonzero real number b, as the functions bx and x show. Thus, answer choices (A), (B), and (C) are incorrect and the correct answer is (D).

18. In this question, you are given a function, N, that models the production of a certain chemical reaction in grams as a function of time, t, in hours. You are asked to find the rate of production at 30 minutes after the reaction begins. The rate of production will be equal to the first derivative of N evaluated at 30 minutes. You should recognize that you first need to convert 30 minutes into hours and then evaluate the first derivative of N at that value of t. Since

30 minutes equals $\frac{1}{2}$ hour, you will need to evaluate $N'\left(\frac{1}{2}\right)$.

First find $N'(t)$. $N'(t) = 16 - 8t$. Therefore,

$N'\left(\frac{1}{2}\right) = 16 - 8\left(\frac{1}{2}\right) = 12$. The answer is 12 grams per hour, so the correct answer is (B).

19. In this question, you will need to use your knowledge of a normally distributed set of data. In particular, you should know that approximately 68 percent of a normally distributed set of data lie within ± 1 standard deviation of the mean and that approximately 95 percent of the data lie within ± 2 standard deviations of the mean. The question asks you to identify which of the groups given in the answer choices is expected to correspond to the greatest number of ninth graders if the hand spans of ninth graders are approximately normally distributed with a mean of 7 inches and a standard deviation of 1 inch. You will need to evaluate each answer choice in order to determine which of the groups is largest.

Answer choice (A) is the group of hand spans less than 6 inches. Since the mean hand span is 7 inches and the standard deviation is 1 inch, the group of hand spans that is less than 6 inches is the group that is more than 1 standard deviation less than the mean. The group of hand spans that is less than 7 inches includes 50 percent of the measurements. Approximately 34 percent ($\frac{1}{2}$ of 68 percent) of the measurements are between 6 inches and 7 inches (within 1 standard deviation less than the mean). So the group with hand spans less than 6 inches would be approximately equal to $50 - 34$, or 16 percent of the measurements.

Answer choice (B) is the group of hand spans greater than 7 inches. Since 7 inches is the mean, approximately 50 percent of the measurements are greater than the mean.

Answer choice (C) is the group of hand spans between 6 and 8 inches. This is the group that is within ± 1 standard deviation of the mean. This group contains approximately 68 percent of the measurements.

Answer choice (D) is the group of hand spans between 5 and 7 inches. This group is between the mean and 2 standard deviations less than the mean. Approximately 47.5 percent ($\frac{1}{2}$ of 95 percent) of the measurements are between 5 inches and 7 inches.

Of the answer choices given, the group described in (C) is expected to contain the greatest percent of the measurements, approximately 68 percent, and would correspond to the largest number of ninth graders, so (C) is the correct answer.

20. A stem plot such as the one shown in this question is a very useful way to display data such as these when you are interested in determining the median value of the data. The data in a stem plot is ordered, so finding the median, the middle number when the data are ordered from least to greatest or greatest to least, is straightforward. You are given the course grades received by 22 students. The median course grade would be the average of the course grades of the 11th and 12th students. You can start at either the least or greatest data entry and count in increasing (or decreasing) order along the leaves until you reach the 11th and 12th entries. In this case, both the 11th and 12th entries have a value of 82 (i.e., a stem value of 8 and a leaf value of 2). Therefore the median course grade received by the 22 students is 82. The correct answer is (D).

21. In this question, you are asked to apply your knowledge of independent events to find the probability of tossing tails exactly 2 out of 3 times when using an unfairly weighted coin. Because each toss of the coin is an independent event, the probability of tossing heads then 2 tails, $P(HTT)$, is equal to $P(H) \cdot P(T) \cdot P(T)$, where $P(H)$ is the probability of tossing heads and $P(T)$ is the probability of tossing tails. In this case, you are given that the probability of tossing heads is twice the probability of tossing tails. So, $P(H) = \frac{2}{3}$ and $P(T) = \frac{1}{3}$. (Out of 3 tosses, 2 would be expected to be heads and 1 would expected to be tails.) Therefore $P(HTT) = \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{2}{27}$. There are 3 ways in which exactly 2 of 3 tosses would be tails and each of them has an equal probability of occurring: $P(THT) = P(TTH) = P(HTT) = \frac{2}{27}$. Therefore the total probability that tails will result exactly 2 times in 3 tosses is $3\left(\frac{2}{27}\right) = \frac{2}{9}$. So the correct answer is (A).

22. In order to answer this question, you need to consider how matrix multiplication is performed. You are asked to find a matrix, M, that when multiplied by any matrix of the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, yields the result $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$. You will notice that all of the answer choices are 3×3 matrices. You can either solve this problem for the general case or reason to the answer. First, the general solution: Let $M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$. Then

$$M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax + by + cz \\ dx + ey + fz \\ gx + hy + jz \end{pmatrix}$$

for all $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$. Since $M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ for all $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then

$$\begin{cases} ax + by + cz = x \\ dx + ey + fz = y \\ gx + hy + jz = 0 \end{cases}$$ for all x, y, and z. This implies $a = 1,\ b = 0,\ c = 0$; and $d = 0,\ e = 1,\ f = 0$; and $g = h = j = 0$;

and therefore $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The correct answer is (B).

You could also reason to the answer by inspecting the answer choices given. Since multiplying the first row of M by the matrix $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ has to result in only the x term for all x, y, and z, the first entry in the first row must be 1 and the others 0. Likewise, multiplying the second row of M by the matrix $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ will result only in the y term for all x, y, and z, so the entries in the second row must be $0, 1, 0$, in that order. Multiplying the third row of M by the matrix $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ results in 0 for all x, y, and z, so the entries in the third row must all be 0. Therefore $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and the correct answer is (B).

23. This question asks you to find the value of x for which the given matrix is NOT invertible. A matrix is not invertible if the determinant of the matrix is equal to zero. The determinant of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is equal to $ad - bc$. For the matrix given in the question, the determinant is equal to $(1)(6) - (4)(x)$. This equals 0 when $6 - 4x = 0$, or $x = \frac{3}{2}$. The correct answer then is (C).

24. Given the recursive function defined in the question, in order to find $f(4)$, you need first to find $f(2)$ and $f(3)$. ($f(1)$ is given.)

Since $f(1) = -3$ and $f(n) = f(n-1) - 6$ for $n \geq 2$, then

$$f(2) = -3 - 6 = -9$$
$$f(3) = -9 - 6 = -15$$
$$f(4) = -15 - 6 = -21$$

So the correct answer is (D).

25. To answer this question, you must read each answer choice and find the statement that correctly describes the properties of the relation defined as "is perpendicular to." You can see that each answer choice includes two of three properties: reflexivity, symmetry, or transitivity. It may be most efficient to consider each of these properties first and then find the statement that describes these properties correctly for the given relation. The definition of these properties can be found in the Notation, Definitions, and Formulas pages that are included in this document and at the beginning of each of the Mathematics: Content Knowledge tests.

A relation $\Re$ is reflexive if $x \Re x$ for all x. In this case, a line cannot be perpendicular to itself, so the relation given in the question is not reflexive.

A relation $\Re$ is symmetric if $x \Re y \Rightarrow y \Re x$ for all x and y. In this case, if line j is perpendicular to line k, it follows that line k is perpendicular to line j. So this relation is symmetric.

A relation $\Re$ is transitive if $(x \Re y \text{ and } y \Re z) \Rightarrow x \Re z$ for all x, y, and z. In this case, if line j is perpendicular to line k and line k is perpendicular to line l, then lines j and l are either the same line or are parallel to each other. Thus, line j is not perpendicular to line l. So this relation is not transitive.

The answer choice that correctly describes the relation "is perpendicular to" is (B), "symmetric but not transitive."

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