Chapter 8

Sequences, L'Hôpital's Rule, and Improper Integrals

NASA's Mars Pathfinder collected and transmitted scientific data and photographs back to Earth.

How much work must be done against gravity for the 2000-pound Pathfinder to escape Earth's gravity, that is, to be lifted an infinite distance above the surface of Earth? Assume that the force due to gravity on an object of weight w, r miles from the center of Earth is:

$$F = 16{,}000{,}000\, w/(r^2) \quad (r \geq 4000) \text{ (in pounds).}$$

The radius of Earth is approximately 4000 miles. The concepts in Section 8.4 will help you solve this problem.

Chapter 8 Overview

In the late seventeenth century, John Bernoulli discovered a rule for calculating limits of fractions whose numerators and denominators both approach zero. The rule is known today as l'Hôpital's Rule, after Guillaume François Antoine de l'Hôpital (1661–1704), Marquis de St. Mesme, a French nobleman who wrote the first differential calculus text, where the rule first appeared in print. We will also use l'Hôpital's Rule to compare the rates at which functions of x grow as $|x|$ becomes large.

In Chapter 5 we saw how to evaluate definite integrals of continuous functions and bounded functions with a finite number of discontinuities on finite closed intervals. These ideas are extended to integrals where one or both limits of integration are infinite, and to integrals whose integrands become unbounded on the interval of integration. Sequences are introduced in preparation for the study of infinite series in Chapter 9.

8.1 Sequences

What you'll learn about

- Defining a Sequence
- Arithmetic and Geometric Sequences
- Graphing a Sequence
- Limit of a Sequence

. . . and why

Sequences arise frequently in mathematics and applied fields.

Defining a Sequence

We have seen sequences before, such as sequences $x_0, x_1, \dots, x_n, \dots$ of numerical approximations generated by Newton's method in Chapter 4. A **sequence** $\{a_n\}$ is a list of numbers written in an explicit order. For example, in the sequence

$$\{a_n\} = \{a_1, a_2, a_3, \dots, a_n, \dots\},$$

a_1 is the *first term,* a_2 is the *second term,* a_3 is the *third term,* and so forth. The numbers $a_1, a_2, a_3, \dots, a_n, \dots$ are the **terms** of the sequence and a_n is the **nth term** of the sequence. We may also think of the sequence $\{a_1, a_2, a_3, \dots, a_n, \dots\}$ as a function with domain the set of positive integers and range $\{a_1, a_2, a_3, \dots, a_n, \dots\}$.

Any real-valued function with domain a subset of the set of positive integers is considered a sequence. If the domain is finite, then the sequence is a **finite sequence**. Generally we will concentrate on **infinite sequences**, that is, sequences with domains that are infinite subsets of the positive integers.

EXAMPLE 1 Defining a Sequence Explicitly

Find the first six terms and the 100th term of the sequence $\{a_n\}$ where

$$a_n = \frac{(-1)^n}{n^2+1}.$$

SOLUTION

Set n equal to 1, 2, 3, 4, 5, 6, and we obtain

$$a_1 = \frac{(-1)^1}{1^2+1} = -\frac{1}{2},\quad a_2 = \frac{(-1)^2}{2^2+1} = \frac{1}{5},\quad a_3 = -\frac{1}{10},\quad a_4 = \frac{1}{17},\quad a_5 = -\frac{1}{26},\quad a_6 = \frac{1}{37}.$$

For $n = 100$ we find

$$a_{100} = \frac{1}{100^2+1} = \frac{1}{10001}.$$

Now try Exercise 1.

The sequence of Example 1 is defined **explicitly** because the formula for a_n is defined in terms of n. Another way to define a sequence is **recursively** by giving a formula for a_n relating it to previous terms, as shown in Example 2.

EXAMPLE 2 Defining a Sequence Recursively

Find the first four terms and the eighth term for the sequence defined recursively by the conditions:

$$b_1 = 4$$
$$b_n = b_{n-1} + 2 \text{ for all } n \geq 2.$$

SOLUTION

We proceed one term at a time, starting with $b_1 = 4$ and obtaining each succeeding term by adding 2 to the term just before it:

$$b_1 = 4$$
$$b_2 = b_1 + 2 = 6$$
$$b_3 = b_2 + 2 = 8$$
$$b_4 = b_3 + 2 = 10$$

and so forth.

Continuing in this way we arrive at $b_8 = 18$. *Now try Exercise 5.*

Arithmetic and Geometric Sequences

There are a variety of rules by which we can construct sequences, but two particular types of sequence are dominant in mathematical applications: those in which pairs of successive terms all have a *common difference* (*arithmetic sequences*), and those in which pairs of successive terms all have a *common quotient,* or *common ratio* (*geometric sequences*).

DEFINITION Arithmetic Sequence

A sequence $\{a_n\}$ is an **arithmetic sequence** if it can be written in the form

$$\{a, a + d, a + 2d, \ldots, a + (n - 1)d, \ldots\}$$

for some constant d. The number d is the **common difference**.

Each term in an arithmetic sequence can be obtained recursively from its preceding term by adding d:

$$a_n = a_{n-1} + d \quad \text{for all} \quad n \geq 2.$$

EXAMPLE 3 Defining Arithmetic Sequences

For each of the following arithmetic sequences, find **(a)** the common difference, **(b)** the ninth term, **(c)** a recursive rule for the nth term, and **(d)** an explicit rule for the nth term.

Sequence 1: $-5, -2, 1, 4, 7, \ldots$ **Sequence 2:** $\ln 2, \ln 6, \ln 18, \ln 54, \ldots$

SOLUTION

Sequence 1

(a) The difference between successive terms is 3.

(b) $a_9 = -5 + (9 - 1)(3) = 19$

(c) The sequence is defined recursively by $a_1 = -5$ and $a_n = a_{n-1} + 3$ for all $n \geq 2$.

(d) The sequence is defined explicitly by $a_n = -5 + (n - 1)(3) = 3n - 8$.

Sequence 2

(a) The difference between the first two terms is $\ln 6 - \ln 2 = \ln (6/2) = \ln 3$. You can check that $\ln 18 - \ln 6 = \ln 54 - \ln 18$ are also equal to $\ln 3$.

continued

(b) $a_9 = \ln 2 + (9 - 1)(\ln 3) = \ln 2 + 8 \ln 3 = \ln (2 \cdot 3^8) = \ln 13{,}122$

(c) The sequence is defined recursively by $a_1 = \ln 2$ and $a_n = a_{n-1} + \ln 3$ for all $n \geq 2$.

(d) The sequence is defined explicitly by $a_n = \ln 2 + (n - 1)(\ln 3) = \ln (2 \cdot 3^{n-1})$.

Now try Exercise 13.

DEFINITION Geometric Sequence

A sequence $\{a_n\}$ is an **geometric sequence** if it can be written in the form

$$\{a, a \cdot r, a \cdot r^2, \dots, a \cdot r^{n-1}, \dots\}$$

for some nonzero constant r. The number r is the **common ratio**.

Each term in a geometric sequence can be obtained recursively from its preceding term by multiplying by r:

$$a_n = a_{n-1} \cdot r \quad \text{for all} \quad n \geq 2.$$

EXAMPLE 4 Defining Geometric Sequences

For each of the following geometric sequences, find **(a)** the common ratio, **(b)** the tenth term, **(c)** a recursive rule for the nth term, and **(d)** an explicit rule for the nth term.

Sequence 1: $1, -2, 4, -8, 16, \dots$ **Sequence 2:** $10^{-2}, 10^{-1}, 1, 10, 10^2, \dots$

SOLUTION

Sequence 1

(a) The ratio between successive terms is -2.

(b) $a_{10} = (1) \cdot (-2)^9 = -512$

(c) The sequence is defined recursively by $a_1 = 1$ and $a_n = (-2)a_{n-1}$ for $n \geq 2$.

(d) The sequence is defined explicitly by $a_n = (1) \cdot (-2)^{n-1} = (-2)^{n-1}$.

Sequence 2

(a) The ratio between successive terms is 10.

(b) $a_{10} = (10^{-2}) \cdot (10^9) = 10^7$

(c) The sequence is defined recursively by $a_1 = 10^{-2}$ and $a_n = (10)a_{n-1}$ for $n \geq 2$.

(d) The sequence is defined explicitly by $a_n = (10^{-2}) \cdot (10^{n-1}) = 10^{n-3}$.

Now try Exercise 17.

EXAMPLE 5 Constructing a Sequence

The second and fifth terms of a geometric sequence are 6 and -48, respectively. Find the first term, common ratio, and an explicit rule for the nth term.

SOLUTION

Because the sequence is geometric the second term is $a_1 \cdot r$ and the fifth term is $a_1 \cdot r^4$, where a_1 is the first term and r is the common ratio. Dividing, we have

$$\frac{a_1 \cdot r^4}{a_1 \cdot r} = -\frac{48}{6}$$

$$r^3 = -8$$

$$r = -2.$$

continued

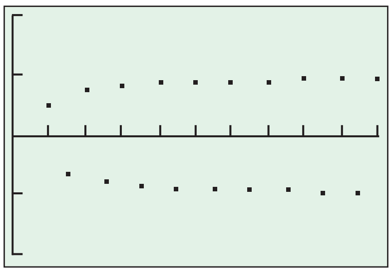
Then $a_1 \cdot r = 6$ implies that $a_1 = -3$. The sequence is defined explicitly by

$$a_n = (-3)(-2)^{n-1} = (-1)^n(3)(2^{n-1}).$$

Now try Exercise 21.

Graphing a Sequence

As with other kinds of functions, it helps to represent a sequence geometrically with its graph. One way to produce a graph of a sequence on a graphing calculator is to use parametric mode, as shown in Example 6.



[0, 20] by [–2, 2]

Figure 8.1 The sequence of Example 6.

EXAMPLE 6 Graphing a Sequence Using Parametric Mode

Draw a graph of the sequence $\{a_n\}$ with $a_n = (-1)^n \dfrac{n-1}{n}$, $n = 1, 2, \ldots$.

SOLUTION

Let $X_{1T} = T$, $Y_{1T} = (-1)^T \dfrac{T-1}{T}$, and graph in dot mode. Set Tmin = 1, Tmax = 20, and Tstep = 1. Even through the domain of the sequence is all positive integers, we are required to choose a value for Tmax to use parametric graphing mode. Finally, we choose Xmin = 0, Xmax = 20, Xscl = 2, Ymin = −2, Ymax = 2, Yscl = 1, and draw the graph (Figure 8.1). ***Now try Exercise 23.***

Some graphing calculators have a built-in sequence graphing mode that makes it easy to graph sequences defined recursively. The function names used in this mode are u, v, and w. We will use this procedure to graph the sequence of Example 7.

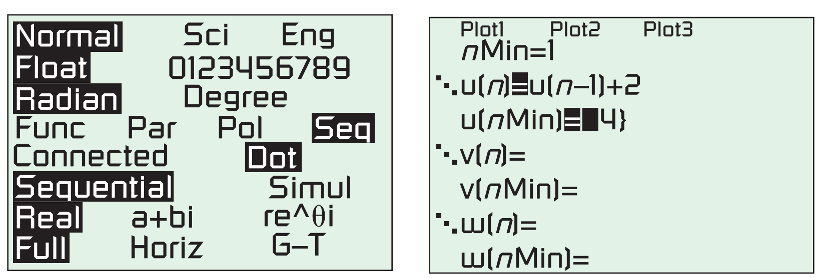
EXAMPLE 7 Graphing a Sequence Using Sequence Graphing Mode

Graph the sequence defined recursively by

$$b_1 = 4$$
$$b_n = b_{n-1} + 2 \quad \text{for all} \quad n \geq 2.$$

SOLUTION

We set the calculator in Sequence graphing mode and dot mode (Figure 8.2a). Replace b_n by u(n). Then select nMin = 1, u(n) = u(n − 1) + 2, and u(nMin) = {4} (Figure 8.2b).

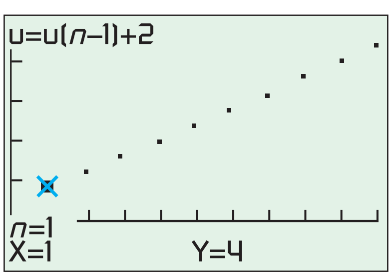


(a) (b)

Figure 8.2 (a) Setting sequence mode and dot mode on the calculator, and (b) entering the sequence of Example 7 in the calculator.

Then set nMin = 1, nMax = 10, PlotStart = 1, PlotStep = 1, and graph in the [0, 10] by [−5, 25] viewing window (Figure 8.3). We have also activated Trace in Figure 8.3.

Now try Exercise 27.



[0, 10] by [–5, 25]

Figure 8.3 The graph of the sequence of Example 7. The TRACE feature shows the coordinates of the first point (1, 4) of the sequence,

$$b_1 = 4, b_n = b_{n-1} + 2, n \geq 2.$$

Limit of a Sequence

The sequence $\{1, 2, 3, \dots, n, \dots\}$ of positive integers has no limit. As with functions, we can use a grapher to suggest what a limiting value may be, and then we can confirm the limit analytically with theorems based on a formal definition as we did in Chapter 2.

DEFINITION Limit

Let L be a real number. The sequence $\{a_n\}$ has **limit L as n approaches ∞** if, given any positive number ϵ, there is a positive number M such that for all $n > M$ we have

$$|a_n - L| < \epsilon.$$

We write $\lim_{n\to\infty} a_n = L$ and say that the sequence **converges to L**. Sequences that do not have limits **diverge**.

Just as in Chapter 2, there are important properties of limits that help us compute limits of sequences.

THEOREM 1 Properties of Limits

If L and M are real numbers and $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then

1. *Sum Rule:* $\lim_{n\to\infty} (a_n + b_n) = L + M$
2. *Difference Rule:* $\lim_{n\to\infty} (a_n - b_n) = L - M$
3. *Product Rule:* $\lim_{n\to\infty} (a_n b_n) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{n\to\infty} (c \cdot a_n) = c \cdot L$
5. *Quotient Rule:* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}, \quad M \neq 0$

EXAMPLE 8 Finding the Limit of a Sequence

Determine whether the sequence converges or diverges. If it converges, find its limit.

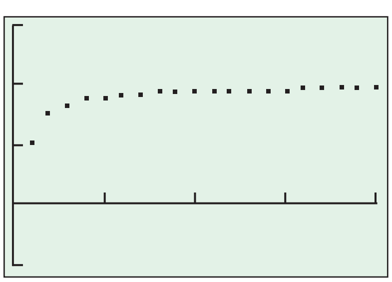
$$a_n = \frac{2n - 1}{n}$$

SOLUTION

It appears from the graph of the sequence in Figure 8.4 that the limit exists. Analytically, using Properties of Limits we have

$$\begin{aligned}\lim_{n\to\infty} \frac{2n-1}{n} &= \lim_{n\to\infty} \left(2 - \frac{1}{n}\right)\\ &= \lim_{n\to\infty} (2) - \lim_{n\to\infty} \left(\frac{1}{n}\right)\\ &= 2 - 0 = 2.\end{aligned}$$

The sequence converges and its limit is 2. *Now try Exercise 31.*



[0, 20] by [–1, 3]

Figure 8.4 The graph of the sequence in Example 8.

EXAMPLE 9 Determining Convergence or Divergence

Determine whether the sequence with given nth term converges or diverges. If it converges, find its limit.

(a) $a_n = (-1)^n \dfrac{n-1}{n}$, $n = 1, 2, \ldots$ **(b)** $b_1 = 4$, $b_n = b_{n-1} + 2$ for all $n \geq 2$

SOLUTION

(a) This is the sequence of Example 6 with graph shown in Figure 8.1. This sequence diverges. In fact we can see that the terms with n even approach 1 while the terms with n odd approach -1.

(b) This is the sequence of Example 7 with graph shown in Figure 8.3. This sequence also diverges. In fact we can say that $\lim\limits_{n\to\infty} b_n = \infty$. ***Now try Exercise 35.***

An important theorem that can be rewritten for sequences is the Sandwich Theorem from Chapter 2.

THEOREM 2 The Sandwich Theorem for Sequences

If $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$ and if there is an integer N for which $a_n \leq b_n \leq c_n$ for all $n > N$, then $\lim\limits_{n\to\infty} b_n = L$.

EXAMPLE 10 Using the Sandwich Theorem

Show that the sequence $\left\{\dfrac{\cos n}{n}\right\}$ converges, and find its limit.

SOLUTION

Because $|\cos x| \leq 1$ for all x, it follows that

$$\left|\frac{\cos n}{n}\right| \leq \frac{|\cos n|}{|n|} \leq \frac{1}{n}$$

for all integers $n \geq 1$. Thus,

$$-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}.$$

Then, $\lim\limits_{n\to\infty} \dfrac{\cos n}{n} = 0$ because $\lim\limits_{n\to\infty}\left(-\dfrac{1}{n}\right) = \lim\limits_{n\to\infty}\left(\dfrac{1}{n}\right) = 0$ and the sequence $\left\{\dfrac{\cos n}{n}\right\}$ converges. ***Now try Exercise 41.***

We can use the Sandwich Theorem to prove the following theorem.

THEOREM 3 Absolute Value Theorem

Consider the sequence $\{a_n\}$. If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

Proof We know that $-|a_n| \leq a_n \leq |a_n|$. Thus, $\lim\limits_{n\to\infty} |a_n| = 0$ and $\lim\limits_{n\to\infty} -|a_n| = 0$ implies that $\lim\limits_{n\to\infty} a_n = 0$ because of the Sandwich Theorem. ■

Another way to state the Absolute Value Theorem is that if the absolute value sequence converges to 0, then the original sequence also converges to 0.

Quick Review 8.1 *(For help, go to Sections 1.2, 2.1, and 2.2.)*

In Exercises 1 and 2, let $f(x) = \dfrac{x}{x+3}$. Find the values of f.

1. $f(5)$ 5/8 **2.** $f(-2)$ -2

In Exercises 3 and 4, evaluate the expression $a + (n-1)d$ for the given values of a, n, and d.

3. $a = -2, n = 3, d = 1.5$ 1

4. $a = -7, n = 5, d = 3$ 5

In Exercises 5 and 6, evaluate the expression ar^{n-1} for the given values of a, r, and n.

5. $a = 1.5, r = 2, n = 4$ 12 **6.** $a = -2, r = 1.5, n = 3$ -4.5

In Exercises 7–10, find the value of the limit.

7. $\lim_{x\to\infty} \dfrac{5x^3 + 2x^2}{3x^4 + 16x^2}$ 0 **8.** $\lim_{x\to 0} \dfrac{\sin(3x)}{x}$ 3

9. $\lim_{x\to\infty} \left(x \sin\left(\dfrac{1}{x}\right)\right)$ 1 **10.** $\lim_{x\to\infty} \dfrac{2x^3 + x^2}{x+1}$ Does not exist, or ∞

Section 8.1 Exercises

In Exercises 1–4, find the first six terms and the 50th term of the sequence with specified nth term.

1. $a_n = \dfrac{n}{n+1}$ See page 443. **2.** $b_n = 3 - \dfrac{1}{n}$ See page 443.

3. $c_n = \left(1 + \dfrac{1}{n}\right)^n$ See page 443. **4.** $d_n = n^2 - 3n$ $-2, -2, 0, 4, 10, 18; 2350$

In Exercises 5–10, find the first four terms and the eighth term of the recursively defined sequence.

5. $a_1 = 3, a_n = a_{n-1} - 2$ for all $n \ge 2$ $3, 1, -1, -3; -11$

6. $b_1 = -2, b_n = b_{n-1} + 1$ for all $n \ge 2$ $-2, -1, 0, 1; 5$

7. $c_1 = 2, c_n = 2c_{n-1}$ for all $n \ge 2$ 2, 4, 8, 16; 256

8. $d_1 = 10, d_n = 1.1d_{n-1}$ for all $n \ge 2$ 10, 11, 12.1, 13.31; 19.487171 = $10(1.1)^7$

9. $u_1 = 1, u_2 = 1, u_n = u_{n-1} + u_{n-2}$ for all $n \ge 3$ 1, 1, 2, 3; 21

10. $v_1 = -3, v_2 = 2, v_n = v_{n-1} + v_{n-2}$ for all $n \ge 3$ $-3, 2, -1, 1; 2$

In Exercises 11–14, the sequences are arithmetic. Find

(a) the common difference,

(b) the eighth term,

(c) a recursive rule for the nth term, and

(d) an explicit rule for the nth term.

11. $-2, 1, 4, 7, \ldots$ See page 443. **12.** $15, 13, 11, 9, \ldots$ See page 443.

13. $1, 3/2, 2, 5/2, \ldots$ See page 443. **14.** $3, 3.1, 3.2, 3.3, \ldots$ See page 443.

In Exercises 15–18, the sequences are geometric. Find

(a) the common ratio,

(b) the ninth term,

(c) a recursive rule for the nth term, and

(d) an explicit rule for the nth term.

15. $8, 4, 2, 1, \ldots$ See page 443. **16.** $1, 1.5, 2.25, 3.375, \ldots$ See page 443.

17. $-3, 9, -27, 81, \ldots$ See page 443. **18.** $5, -5, 5, -5, \ldots$ See page 443.

19. The second and fifth terms of an arithmetic sequence are -2 and 7, respectively. Find the first term and a recursive rule for the nth term. -5, $a_n = a_{n-1} + 3$ for all $n \ge 2$

20. The fifth and ninth terms of an arithmetic sequence are 5 and -3, respectively. Find the first term and an explicit rule for the nth term. 13, $a_n = -2n + 15, n \ge 1$

21. The fourth and seventh terms of a geometric sequence are 3010 and 3,010,000, respectively. Find the first term, common ratio, and an explicit rule for the nth term. $a_1 = 3.01, r = 10,$ $a_n = 3.01(10)^{n-1}, n \ge 1$

22. The second and seventh terms of a geometric sequence are $-1/2$ and 16, respectively. Find the first term, common ratio, and an explicit rule for the nth term. $a_1 = 1/4, r = -2,$ $a_n = (-1)^{n-1}(2)^{n-3}, n \ge 1$

In Exercises 23–30, draw a graph of the sequence $\{a_n\}$.

23. $a_n = \dfrac{n}{n^2+1}, \quad n = 1, 2, 3, \ldots$

24. $a_n = \dfrac{n-2}{n+2}, \quad n = 1, 2, 3, \ldots$

25. $a_n = (-1)^n \dfrac{2n+1}{n}, \quad n = 1, 2, 3, \ldots$

26. $a_n = \left(1 + \dfrac{2}{n}\right)^n, n = 1, 2, 3, \ldots$

27. $u_1 = 2, \quad u_n = 3u_{n-1}$ for all $n \ge 2$

28. $u_1 = 2, \quad u_n = u_{n-1} + 3$ for all $n \ge 2$

29. $u_1 = 3, \quad u_n = 5 - \dfrac{1}{2}u_{n-1}$ for all $n \ge 2$

30. $u_1 = 5, \quad u_n = u_{n-1} - 2$ for all $n \ge 2$

In Exercises 31–40, determine the convergence or divergence of the sequence with given nth term. If the sequence converges, find its limit.

31. $a_n = \dfrac{3n+1}{n}$ converges, 3 **32.** $a_n = \dfrac{2n}{n+3}$ converges, 2

33. $a_n = \dfrac{2n^2 - n - 1}{5n^2 + n + 2}$ converges, 2/5 **34.** $a_n = \dfrac{n}{n^2+1}$ converges, 0

35. $a_n = (-1)^n \dfrac{n-1}{n+3}$ diverges **36.** $a_n = (-1)^n \dfrac{n+1}{n^2+2}$ converges, 0

37. $a_n = (1.1)^n$ diverges **38.** $a_n = (0.9)^n$ converges, 0

39. $a_n = n \sin\left(\dfrac{1}{n}\right)$ converges, 1 **40.** $a_n = \cos\left(n\dfrac{\pi}{2}\right)$ diverges

In Exercises 41–44, use the Sandwich Theorem to show that the sequence with given nth term converges and find its limit.

41. $a_n = \dfrac{\sin n}{n}$ 0 **42.** $a_n = \dfrac{1}{2^n}$ 0 (Note: $\dfrac{1}{2^n} < \dfrac{1}{n}$ for $n \ge 1$)

43. $a_n = \dfrac{1}{n!}$ **44.** $a_n = \dfrac{\sin^2 n}{2^n}$ 0

0, (Note: $\dfrac{1}{n!} \le \dfrac{1}{n}$ for $n \ge 1$)

In Exercises 45–48, match the graph or table with the sequence with given nth term.

45. $a_n = \dfrac{2n - 1}{n}$ Graph (b)

46. $b_n = (-1)^n \dfrac{3n + 1}{n + 3}$ Graph (c)

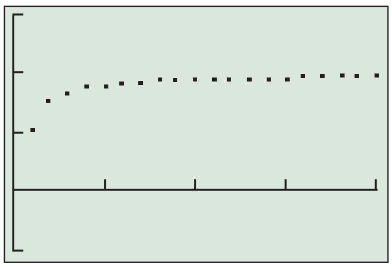
47. $c_n = \dfrac{n + 1}{n}$ Table (d)

48. $d_n = \dfrac{4}{n + 2}$ Table (a)

n	u(n)
1	1.3333
2	1
3	.8
4	.66667
5	.57143
6	.5
7	.44444

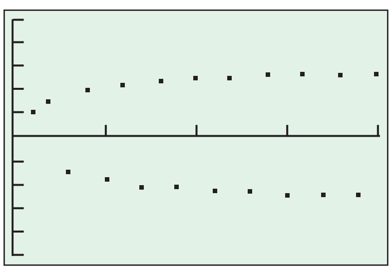
n = 1

(a)



[0, 20] by [–1, 3]

(b)



[0, 20] by [–5, 5]

(c)

n	u(n)
1	2
2	1.5
3	1.3333
4	1.25
5	1.2
6	1.1667
7	1.1429

n = 1

(d)

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

49. True or False If the first two terms of an arithmetic sequence are negative, then all its terms are negative. Justify your answer.

50. True or False If the first two terms of a geometric sequence are positive, then all its terms are positive. Justify your answer.

51. Multiple Choice The first and third terms of an arithmetic sequence are -1 and 5, respectively. Which of the following is the sixth term? C

(A) -25 **(B)** 11 **(C)** 14 **(D)** 29 **(E)** 3125

52. Multiple Choice The second and third terms of a geometric sequence are 2.5 and 1.25, respectively. Which of the following is the first term? E

(A) -5 **(B)** -2.5 **(C)** 0.625 **(D)** 3.75 **(E)** 5

53. Multiple Choice Which of the following is the limit of the sequence with nth term $a_n = n \sin\left(\dfrac{3\pi}{n}\right)$? D

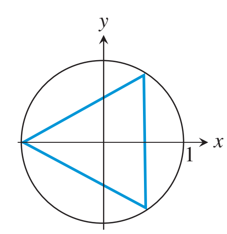
(A) 1 **(B)** π **(C)** 2π **(D)** 3π **(E)** 4π

54. Multiple Choice Which of the following is the limit of the sequence with nth term $a_n = (-1)^n \dfrac{3n - 1}{n + 2}$? E

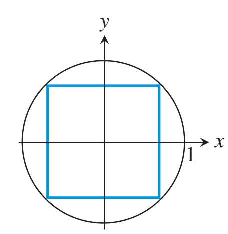
(A) -3 **(B)** 0 **(C)** 2 **(D)** 3 **(E)** diverges

Explorations

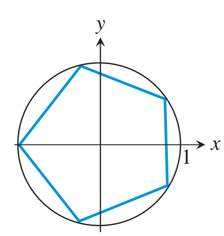
55. *Connecting Geometry and Sequences* In the sequence of diagrams that follow, regular polygons are inscribed in unit circles with at least one side of each polygon perpendicular to the x-axis.



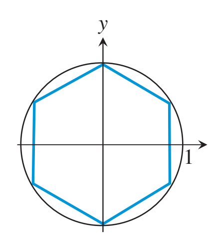
(a)



(b)



(c)



(d)

(a) Prove that the perimeter of each polygon in the sequence is given by $a_n = 2n \sin(\pi/n)$, where n is the number of sides in the polygon.

(b) Determine $\lim_{n\to\infty} a_n$. 2π

49. False. Consider the sequence with nth term $a_n = -5 + 2(n - 1)$. Here $a_1 = -5$, $a_2 = -3$, $a_3 = -1$, and $a_4 = 1$.

50. True. $a_1 > 0$, $r = a_2/a_1 > 0$, and $a_n = a_1 r^{n-1} > 0$ for all $n \geq 2$.

56. ***Fibonacci Sequence*** The **Fibonacci Sequence** can be defined recursively by $a_1 = 1$, $a_2 = 1$, and $a_n = a_{n-2} + a_{n-1}$ for all integers $n \geq 3$.

(a) Write out the first 10 terms of the sequence.
1, 1, 2, 3, 5, 8, 13, 21, 34, 55

(b) Draw a graph of the sequence using the Sequence Graphing mode on your grapher. Enter u(n) = u($n - 1$) + u($n - 2$) and u(nMin) = {1, 1}.

Extending the Ideas

57. Writing to Learn If $\{a_n\}$ is a geometric sequence with all positive terms, explain why $\{\log a_n\}$ must be arithmetic.

58. Writing to Learn If $\{a_n\}$ is an arithmetic sequence, explain why $\{10^{a_n}\}$ must be geometric.

59. ***Proving Limits*** Use the formal definition of limit to prove that $\lim_{n \to \infty} \frac{1}{n} = 0$.

1. 1/2, 2/3, 3/4, 4/5, 5/6, 6/7; 50/51
2. 2, 5/2, 8/3, 11/4, 14/5, 17/6; 149/50
3. 2, 9/4, 64/27, 625/256, 7776/3125 ≈ 2.48832, 117649/46656 ≈ 2.521626; $(51/50)^{50} \approx 2.691588$

11. (a) 3 **(b)** 19
(c) $a_n = a_{n-1} + 3$ **(d)** $a_n = 3n - 5$
12. (a) –2 **(b)** 1
(c) $a_n = a_{n-1} - 2$ **(d)** $a_n = -2n + 17$
13. (a) 1/2 **(b)** 9/2
(c) $a_n = a_{n-1} + 1/2$ **(d)** $a_n = (n + 1)/2$
14. (a) 0.1 **(b)** 3.7
(c) $a_n = a_{n-1} + 0.1$ **(d)** $a_n = 0.1n + 2.9$
15. (a) 1/2 **(b)** $8(1/2)^8 = 0.03125$
(c) $a_n = (1/2)a_{n-1}$ **(d)** $a_n = 8(1/2)^{n-1} = 2^{4-n}$
16. (a) 1.5 **(b)** $(1)(1.5)^8 \approx 25.6289$
(c) $a_n = (1.5)a_{n-1}$ **(d)** $a_n = (1)(1.5)^{n-1} = (1.5)^{n-1}$
17. (a) –3 **(b)** $(-3)^9 = -19{,}683$
(c) $a_n = (-3)a_{n-1}$ **(d)** $a_n = (-3)(-3)^{n-1} = (-3)^n$
18. (a) –1 **(b)** $(5)(-1)^8 = 5$
(c) $a_n = -a_{n-1}$ **(d)** $a_n = 5(-1)^{n-1}$

57. $a_n = ar^{n-1}$ implies that $\log a_n = \log a + (n - 1)\log r$. Thus $\{\log a_n\}$ is an arithmetic sequence with first term $\log a$ and common ratio $\log r$.
58. $a_n = a + (n - 1)d$ implies that $10^{a_n} = 10^{a+(n-1)d} = 10^a(10^d)^{n-1}$. Thus $\{10^{a_n}\}$ is a geometric sequence with first term 10^a and common ratio 10^d.
59. Given $\epsilon > 0$ choose $M = 1/\epsilon$. Then $\left|\frac{1}{n} - 0\right| < \epsilon$ if $n > M$.

8.2 L'Hôpital's Rule

What you'll learn about

- Indeterminate Form 0/0
- Indeterminate Forms ∞/∞, $\infty \cdot 0$, $\infty - \infty$
- Indeterminate Forms 1^∞, 0^0, ∞^0

. . . and why

Limits can be used to describe the behavior of functions and l'Hôpital's Rule is an important technique for finding limits.

Indeterminate Form 0/0

If functions $f(x)$ and $g(x)$ are both zero at $x = a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be found by substituting $x = a$. The substitution produces 0/0, a meaningless expression known as an **indeterminate form.** Our experience so far has been that limits that lead to indeterminate forms may or may not be hard to find algebraically. It took a lot of analysis in Exercise 75 of Section 2.1 to find $\lim_{x \to 0} (\sin x)/x$. But we have had remarkable success with the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

from which we calculate derivatives and which always produces the equivalent of 0/0. L'Hôpital's Rule enables us to draw on our success with derivatives to evaluate limits that otherwise lead to indeterminate forms.

THEOREM 4 L'Hôpital's Rule (First Form)

Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

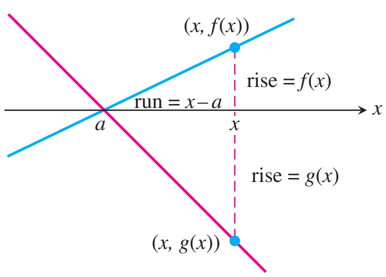


Figure 8.5 A zoom-in view of the graphs of the differentiable functions f and g at $x = a$. (Theorem 4)

Proof

Graphical Argument

If we zoom in on the graphs of f and g at $(a, f(a)) = (a, g(a)) = (a, 0)$, the graphs (Figure 8.5) appear to be straight lines because differentiable functions are locally linear. Let m_1 and m_2 be the slopes of the lines for f and g, respectively. Then for x near a,

$$\frac{f(x)}{g(x)} = \frac{\dfrac{f(x)}{x-a}}{\dfrac{g(x)}{x-a}} = \frac{m_1}{m_2}.$$

As $x \to a$, m_1 and m_2 approach $f'(a)$ and $g'(a)$, respectively. Therefore,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{m_1}{m_2} = \frac{f'(a)}{g'(a)}.$$

Confirm Analytically

Working backward from $f'(a)$ and $g'(a)$, which are themselves limits, we have

$$\frac{f'(a)}{g'(a)} = \frac{\displaystyle\lim_{x \to a} \frac{f(x) - f(a)}{x - a}}{\displaystyle\lim_{x \to a} \frac{g(x) - g(a)}{x - a}} = \lim_{x \to a} \frac{\dfrac{f(x) - f(a)}{x - a}}{\dfrac{g(x) - g(a)}{x - a}}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{f(x) - 0}{g(x) - 0} = \lim_{x \to a} \frac{f(x)}{g(x)}. \quad \blacksquare$$

Bernard A. Harris Jr.
(1956–)

Bernard Harris, M.D., became an astronaut in 1991. In 1995, as the Payload Commander on the STS-63 mission, he became the first African American to walk in space. During this mission, which included a rendezvous with the Russian Space Station, Mir, Harris traveled over 2.9 million miles. Dr. Harris left NASA in 1996 to become Vice President of Microgravity and Life Sciences for SPACEHAB Incorporated.

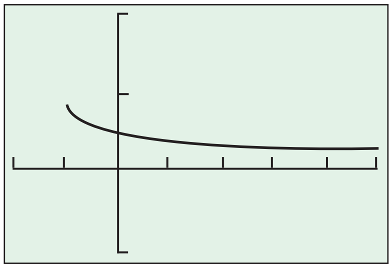


Figure 8.6 The graph of $y = \dfrac{\sqrt{1+x}-1}{x}$. (Example 1)

EXAMPLE 1 Indeterminate Form 0/0

Estimate the limit graphically and then use l'Hôpital's Rule to find the limit.

$$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x}$$

SOLUTION

From the graph in Figure 8.6 we can estimate the limit to be about 1/2. If we set $f(x) = \sqrt{1+x} - 1$ and $g(x) = x$ we have $f(0) = g(0) = 0$. Thus, l'Hôpital's Rule applies in this case. Because $f'(x) = (1/2)(1+x)^{-1/2}$ and $g'(x) = 1$ it follows that

$$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0} \frac{(1/2)(1+x)^{-1/2}}{1} = \frac{1}{2}.$$

Now try Exercise 3.

Sometimes after differentiation the new numerator and denominator both equal zero at $x = a$, as we will see in Example 2. In these cases we apply a stronger form of l'Hôpital's Rule.

Augustin-Louis Cauchy
(1789–1857)

An engineer with a genius for mathematics and mathematical modeling, Cauchy created an early modeling of surface wave propagation that is now a classic in hydrodynamics. Cauchy (pronounced "CO-she") invented our notion of continuity and proved the Intermediate Value Theorem for continuous functions. He invented modern limit notation and was the first to prove the convergence of $(1 + 1/n)^n$. His mean value theorem, the subject of Exercise 71, is the key to proving the stronger form of l'Hôpital's Rule. His work advanced not only calculus and mathematical analysis, but also the fields of complex function theory, error theory, differential equations, and celestial mechanics.

THEOREM 5 L'Hôpital's Rule (Stronger Form)

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

if the latter limit exists.

When you apply l'Hôpital's Rule, look for a change from 0/0 into something else. This is where the limit is revealed.

EXAMPLE 2 Applying a Stronger Form of l'Hôpital's Rule

Evaluate $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x}-1-x/2}{x^2}$.

SOLUTION

Substituting $x = 0$ leads to the indeterminate form 0/0 because the numerator and denominator of the fraction are 0 when 0 is substituted for x. So we apply l'Hôpital's Rule.

$$\lim_{x\to 0} \frac{\sqrt{1+x}-1-x/2}{x^2} = \lim_{x\to 0} \frac{(1/2)(1+x)^{-1/2}-1/2}{2x}$$

Differentiate numerator and denominator.

Substituting 0 for x leads to 0 in both the numerator and denominator of the second fraction, so we differentiate again.

$$\lim_{x\to 0} \frac{\sqrt{1+x}-1-x/2}{x^2} = \lim_{x\to 0} \frac{(1/2)(1+x)^{-1/2}-1/2}{2x} = \lim_{x\to 0} \frac{-(1/4)(1+x)^{-3/2}}{2}$$

The third limit in the above line is $-1/8$. Thus,

$$\lim_{x\to 0} \frac{\sqrt{1+x}-1-x/2}{x^2} = -\frac{1}{8}.$$

Now try Exercise 5.

EXPLORATION 1 **Exploring L'Hôpital's Rule Graphically**

Consider the function $f(x) = \dfrac{\sin x}{x}$.

1. Use l'Hôpital's Rule to find $\lim_{x\to 0} f(x)$.
2. Let $y_1 = \sin x,\ y_2 = x,\ y_3 = y_1/y_2,\ y_4 = y_1'/y_2'$. Explain how graphing y_3 and y_4 in the same viewing window provides support for l'Hôpital's Rule in part 1.
3. Let $y_5 = y_3'$. Graph y_3, y_4, and y_5 in the same viewing window. Based on what you see in the viewing window, make a statement about what l'Hôpital's Rule does *not* say.

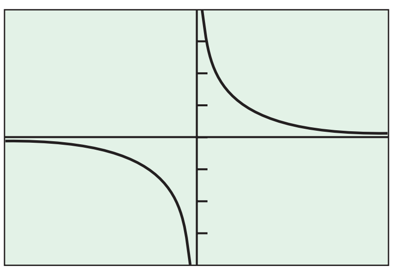
L'Hôpital's Rule applies to one-sided limits as well.

EXAMPLE 3 Using L'Hôpital's Rule with One-Sided Limits

Evaluate the following limits using l'Hôpital's Rule:

(a) $\lim_{x\to 0^+} \dfrac{\sin x}{x^2}$ **(b)** $\lim_{x\to 0^-} \dfrac{\sin x}{x^2}$

Support your answer graphically.



[–1, 1] by [–20, 20]

Figure 8.7 The graph of $f(x) = (\sin x)/x^2$. (Example 3)

SOLUTION

(a) Substituting $x = 0$ leads to the indeterminate form 0/0. Apply l'Hôpital's Rule by differentiating numerator and denominator.

$$\lim_{x\to 0^+} \frac{\sin x}{x^2} = \lim_{x\to 0^+} \frac{\cos x}{2x} \qquad \frac{1}{0}$$
$$= \infty$$

(b) $$\lim_{x\to 0^-} \frac{\sin x}{x^2} = \lim_{x\to 0^-} \frac{\cos x}{2x} \qquad -\frac{1}{0}$$
$$= -\infty$$

Figure 8.7 supports the results. ***Now try Exercise 11.***

When we reach a point where one of the derivatives approaches 0, as in Example 3, and the other does not, then the limit in question is 0 (if the numerator approaches 0) or $\pm$ infinity (if the denominator approaches 0).

Indeterminate Forms ∞/∞, $\infty \cdot 0$, $\infty - \infty$

A version of l'Hôpital's Rule also applies to quotients that lead to the indeterminate form ∞/∞. If $f(x)$ and $g(x)$ both approach infinity as $x \to a$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

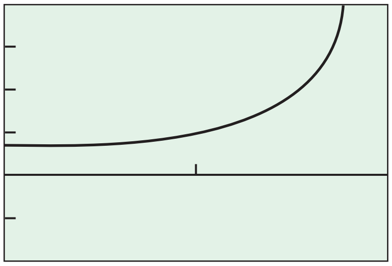
provided the latter limit exists. The a here (and in the indeterminate form 0/0) may itself be finite or infinite, and may be an endpoint of the interval I of Theorem 5.

EXAMPLE 4 Working with Indeterminate Form ∞/∞

Identify the indeterminate form and evaluate the limit using l'Hôpital's Rule. Support your answer graphically.

$$\lim_{x\to \pi/2} \frac{\sec x}{1 + \tan x}$$

continued



[π/4, 3π/4] by [–2, 4]

Figure 8.8 The graph of $y = (\sec x)/(1 + \tan x)$. (Example 4)

SOLUTION

The numerator and denominator are discontinuous at $x = \pi/2$, so we investigate the one-sided limits there. To apply l'Hôpital's Rule we can choose I to be any open interval containing $x = \pi/2$.

$$\lim_{x \to (\pi/2)^-} \frac{\sec x}{1 + \tan x} \qquad \frac{\infty}{\infty} \text{ from the left}$$

Next differentiate the numerator and denominator.

$$\lim_{x \to (\pi/2)^-} \frac{\sec x}{1 + \tan x} = \lim_{x \to (\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x \to (\pi/2)^-} \sin x = 1$$

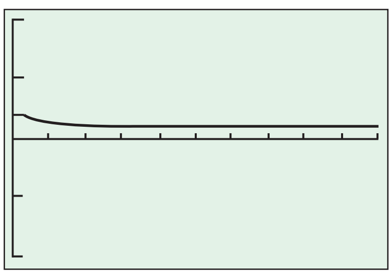
The right-hand limit is 1 also, with $(-\infty)/(-\infty)$ as the indeterminate form. Therefore, the two-sided limit is equal to 1. The graph of $(\sec x)/(1 + \tan x)$ in Figure 8.8 appears to pass right through the point $(\pi/2, 1)$ and supports the work above.

Now try Exercise 13.

EXAMPLE 5 Working with Indeterminate Form ∞/∞

Identify the indeterminate form and evaluate the limit using l'Hôpital's Rule. Support your answer graphically.

$$\lim_{x \to \infty} \frac{\ln x}{2\sqrt{x}}$$



[0, 1000] by [–2, 2]

Figure 8.9 A graph of $y = (\ln x)/(2\sqrt{x})$. (Example 5)

SOLUTION

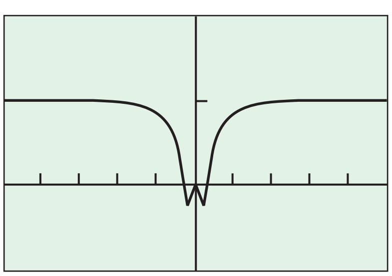
$$\lim_{x \to \infty} \frac{\ln x}{2\sqrt{x}} \overset{\frac{\infty}{\infty}}{=} \lim_{x \to \infty} \frac{1/x}{1/\sqrt{x}} = \lim_{x \to \infty} \frac{1}{\sqrt{x}} = 0$$

The graph in Figure 8.9 supports the result. ***Now try Exercise 15.***

We can sometimes handle the indeterminate forms $\infty \cdot 0$ and $\infty - \infty$ by using algebra to get $0/0$ or ∞/∞ instead. Here again we do not mean to suggest that there is a number $\infty \cdot 0$ or $\infty - \infty$ any more than we mean to suggest that there is a number $0/0$ or ∞/∞. These forms are not numbers but descriptions of function behavior.

EXAMPLE 6 Working With Indeterminate Form $\infty \cdot 0$

Find **(a)** $\displaystyle\lim_{x \to \infty} \left(x \sin \frac{1}{x}\right)$ **(b)** $\displaystyle\lim_{x \to -\infty} \left(x \sin \frac{1}{x}\right)$.



[–5, 5] by [–1, 2]

Figure 8.10 The graph of $y = x \sin (1/x)$. (Example 6)

SOLUTION

Figure 8.10 suggests that the limits exist.

(a)

$$\lim_{x \to \infty} \left(x \sin \frac{1}{x}\right) \qquad \infty \cdot 0$$

$$= \lim_{h \to 0^+} \left(\frac{1}{h} \sin h\right) \qquad \text{Let } h = 1/x.$$

$$= 1$$

(b) Similarly,

$$\lim_{x \to -\infty} \left(x \sin \frac{1}{x}\right) = 1.$$

Now try Exercise 17.

EXAMPLE 7 Working with Indeterminate Form $\infty - \infty$

Find $\displaystyle\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$.

SOLUTION

Combining the two fractions converts the indeterminate form $\infty - \infty$ to $0/0$, to which we can apply l'Hôpital's Rule.

$$\begin{aligned}
&\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) && \infty - \infty \\
&= \lim_{x\to 1}\frac{x - 1 - \ln x}{(x-1)\ln x} && \text{Now } \frac{0}{0} \\
&= \lim_{x\to 1}\frac{1 - 1/x}{\dfrac{x-1}{x} + \ln x} \\
&= \lim_{x\to 1}\frac{x-1}{x\ln x + x - 1} && \text{Still } \frac{0}{0} \\
&= \lim_{x\to 1}\frac{1}{2 + \ln x} \\
&= \frac{1}{2}
\end{aligned}$$

Now try Exercise 19.

Indeterminate Forms 1^∞, 0^0, ∞^0

Limits that lead to the indeterminate forms 1^∞, 0^0, and ∞^0 can sometimes be handled by taking logarithms first. We use l'Hôpital's Rule to find the limit of the logarithm and then exponentiate to reveal the original function's behavior.

Since $b = e^{\ln b}$ for every positive number b, we can write $f(x)$ as

$$f(x) = e^{\ln f(x)}$$

for any positive function $f(x)$.

$$\lim_{x\to a}\ln f(x) = L \quad \Rightarrow \quad \lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^L$$

Here a can be finite or infinite.

In Section 1.3 we used graphs and tables to investigate the values of $f(x) = (1 + 1/x)^x$ as $x\to\infty$. Now we find this limit with l'Hôpital's Rule.

EXAMPLE 8 Working with Indeterminate Form 1^∞

Find $\displaystyle\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x$.

SOLUTION

Let $f(x) = (1 + 1/x)^x$. Then taking logarithms of both sides converts the indeterminate form 1^∞ to $0/0$, to which we can apply l'Hôpital's Rule.

$$\ln f(x) = \ln\left(1 + \frac{1}{x}\right)^x = x\ln\left(1 + \frac{1}{x}\right) = \frac{\ln\left(1 + \dfrac{1}{x}\right)}{\dfrac{1}{x}}$$

continued

We apply l'Hôpital's Rule to the previous expression.

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \qquad \frac{0}{0}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} \qquad \text{Differentiate numerator and denominator.}$$

$$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1$$

Therefore,

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^1 = e.$$

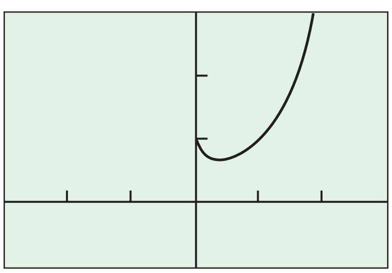
Now try Exercise 21.

EXAMPLE 9 Working with Indeterminate Form 0^0

Determine whether $\lim_{x\to 0^+} x^x$ exists and find its value if it does.

SOLUTION

Investigate Graphically Figure 8.11 suggests that the limit exists and has a value near 1.



[–3, 3] by [–1, 3]

Figure 8.11 The graph of $y = x^x$. (Example 9)

Solve Analytically The limit leads to the indeterminate form 0^0. To convert the problem to one involving 0/0, we let $f(x) = x^x$ and take the logarithm of both sides.

$$\ln f(x) = x \ln x = \frac{\ln x}{1/x}$$

Applying l'Hôpital's Rule to $(\ln x)/(1/x)$ we obtain

$$\lim_{x\to 0^+} \ln f(x) = \lim_{x\to 0^+} \frac{\ln x}{1/x} \qquad \frac{-\infty}{\infty}$$

$$= \lim_{x\to 0^+} \frac{1/x}{-1/x^2} \qquad \text{Differentiate.}$$

$$= \lim_{x\to 0^+} (-x) = 0.$$

Therefore,

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln f(x)} = e^0 = 1.$$

Now try Exercise 23.

EXAMPLE 10 Working with Indeterminate Form ∞^0

Find $\lim_{x\to\infty} x^{1/x}$.

SOLUTION

Let $f(x) = x^{1/x}$. Then

$$\ln f(x) = \frac{\ln x}{x}.$$

continued

Applying l'Hôpital's Rule to $\ln f(x)$ we obtain

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{\ln x}{x} \qquad \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{1/x}{1} \qquad \text{Differentiate.}$$

$$= \lim_{x\to\infty} \frac{1}{x} = 0.$$

Therefore,

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1.$$

Now try Exercise 25.

Quick Review 8.2 *(For help, go to Sections 2.1 and 2.2.)*

In Exercises 1 and 2, use tables to estimate the value of the limit.

1. $\lim_{x\to\infty}\left(1+\frac{0.1}{x}\right)^x$ 1.1052

2. $\lim_{x\to 0^+} x^{1/(\ln x)}$ 2.7183

In Exercises 3–8, use graphs or tables to estimate the value of the limit.

3. $\lim_{x\to 0^-}\left(1-\frac{1}{x}\right)^x$ 1

4. $\lim_{x\to -1^-}\left(1+\frac{1}{x}\right)^x$ ∞

5. $\lim_{t\to 1}\frac{t-1}{\sqrt{t}-1}$ 2

6. $\lim_{x\to\infty}\frac{\sqrt{4x^2+1}}{x+1}$ 2

7. $\lim_{x\to 0}\frac{\sin 3x}{x}$ 3

8. $\lim_{\theta\to\pi/2}\frac{\tan\theta}{2+\tan\theta}$ 1

In Exercises 9 and 10, substitute $x = 1/h$ to express y as a function of h.

9. $y = x\sin\frac{1}{x}$ $y = \frac{\sin h}{h}$

10. $y = \left(1+\frac{1}{x}\right)^x$ $y = (1+h)^{1/h}$

Section 8.2 Exercises

In Exercises 1–4, estimate the limit graphically and then use l'Hôpital's Rule to find the limit.

1. $\lim_{x\to 2}\frac{x-2}{x^2-4}$

2. $\lim_{x\to 0}\frac{\sin(5x)}{x}$

3. $\lim_{x\to 2}\frac{\sqrt{2+x}-2}{x-2}$

4. $\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1}$

In Exercises 5–8, apply the stronger form of l'Hôpital's Rule to find the limit.

5. $\lim_{x\to 0}\frac{1-\cos x}{x^2}$ 1/2

6. $\lim_{\theta\to\pi/2}\frac{1-\sin\theta}{1+\cos(2\theta)}$ 1/4

7. $\lim_{t\to 0}\frac{\cos t-1}{e^t-t-1}$ -1

8. $\lim_{x\to 2}\frac{x^2-4x+4}{x^3-12x+16}$ 1/6

In Exercises 9–12, use l'Hôpital's Rule to evaluate the one-sided limits. Support your answer graphically.

9. (a) $\lim_{x\to 0^-}\frac{\sin 4x}{\sin 2x}$ 2 **(b)** $\lim_{x\to 0^+}\frac{\sin 4x}{\sin 2x}$ 2

10. (a) $\lim_{x\to 0^-}\frac{\tan x}{x}$ 1 **(b)** $\lim_{x\to 0^+}\frac{\tan x}{x}$ 1

11. (a) $\lim_{x\to 0^-}\frac{\sin x}{x^3}$ ∞ **(b)** $\lim_{x\to 0^+}\frac{\sin x}{x^3}$ ∞

12. (a) $\lim_{x\to 0^-}\frac{\tan x}{x^2}$ $-\infty$ **(b)** $\lim_{x\to 0^+}\frac{\tan x}{x^2}$ ∞

In Exercises 13–16, identify the indeterminate form and evaluate the limit using l'Hôpital's Rule. Support your answer graphically.

13. $\lim_{x\to\pi}\frac{\csc x}{1+\cot x}$ Left $(\infty)/(-\infty)$, right $(-\infty)/(\infty)$, limit $= -1$

14. $\lim_{x\to\pi/2}\frac{1+\sec x}{\tan x}$ Left $(\infty)/(\infty)$, right $(-\infty)/(-\infty)$, limit $= 1$

15. $\lim_{x\to\infty}\frac{\ln(x+1)}{\log_2 x}$ $(\infty)/(\infty)$, limit $= \ln 2$

16. $\lim_{x\to\infty}\frac{5x^2-3x}{7x^2+1}$ $(\infty)/(\infty)$, limit $= 5/7$

In Exercises 17–26, identify the indeterminate form and evaluate the limit using l'Hôpital's Rule.

17. $\lim_{x\to 0^+}(x\ln x)$ $\infty\cdot 0$, 0

18. $\lim_{x\to\infty}\left(x\tan\frac{1}{x}\right)$ $\infty\cdot 0$, 1

19. $\lim_{x\to 0^+}(\csc x-\cot x+\cos x)$ $\infty-\infty$, 1

20. $\lim_{x\to\infty}(\ln(2x)-\ln(x+1))$ $\infty-\infty$, $\ln 2$

21. $\lim_{x\to 0}(e^x+x)^{1/x}$ 1^∞, e^2

22. $\lim_{x\to 1}x^{1/(x-1)}$ 1^∞, e

23. $\lim_{x\to 1}(x^2-2x+1)^{x-1}$ 0^0, 1

24. $\lim_{x\to 0^+}(\sin x)^x$ 0^0, 1

25. $\lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x$ ∞^0, 1

26. $\lim_{x\to\infty}(\ln x)^{1/x}$ ∞^0, 1

In Exercises 27 and 28, **(a)** complete the table and estimate the limit. **(b)** Use l'Hôpital's Rule to confirm your estimate.

27. $\lim_{x\to\infty} f(x)$, $f(x) = \frac{\ln x^5}{x}$

x	10	10^2	10^3	10^4	10^5
$f(x)$					

28. $\lim_{x\to 0^+} f(x)$, $f(x) = \frac{x-\sin x}{x^3}$

x	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}
$f(x)$					

62. False. Need $g'(a) \neq 0$. Consider $f(x) = \sin^2 x$ and $g(x) = x^2$ with $a = 0$. Here $\lim_{x\to 0} f'(x) = \lim_{x\to 0} g'(x) = 0$.

In Exercises 29–32, use tables to estimate the limit. Confirm your estimate using l'Hôpital's Rule.

29. $\lim_{\theta\to 0} \dfrac{\sin 3\theta}{\sin 4\theta}$ 3/4

30. $\lim_{t\to 0} \left(\dfrac{1}{\sin t} - \dfrac{1}{t}\right)$ 0

31. $\lim_{x\to\infty} (1 + x)^{1/x}$ 1

32. $\lim_{x\to\infty} \dfrac{x - 2x^2}{3x^2 + 5x}$ $-2/3$

In Exercises 33–52, use l'Hôpital's Rule to evaluate the limit.

33. $\lim_{\theta\to 0} \dfrac{\sin \theta^2}{\theta}$ 0

34. $\lim_{t\to 1} \dfrac{t - 1}{\ln t - \sin \pi t}$ $1/(\pi + 1)$

35. $\lim_{x\to\infty} \dfrac{\log_2 x}{\log_3 (x + 3)}$ ln 3/ln 2

36. $\lim_{y\to 0^+} \dfrac{\ln (y^2 + 2y)}{\ln y}$ 1

37. $\lim_{y\to \pi/2} \left(\dfrac{\pi}{2} - y\right) \tan y$ 1

38. $\lim_{x\to 0^+} (\ln x - \ln \sin x)$ 0

39. $\lim_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{\sqrt{x}}\right)$ ∞

40. $\lim_{x\to 0} \left(\dfrac{1}{x^2}\right)^x$ 1

41. $\lim_{x\to \pm\infty} \dfrac{3x - 5}{2x^2 - x + 2}$ 0

42. $\lim_{x\to 0} \dfrac{\sin 7x}{\tan 11x}$ 7/11

43. $\lim_{x\to\infty} (1 + 2x)^{1/(2 \ln x)}$ $e^{1/2}$

44. $\lim_{x\to(\pi/2)^-} (\cos x)^{\cos x}$ 1

45. $\lim_{x\to 0^+} (1 + x)^{1/x}$ e

46. $\lim_{x\to 0^+} (\sin x)^{\tan x}$ 1

47. $\lim_{x\to 1^+} x^{1/(1-x)}$ e^{-1}

48. $\lim_{x\to\infty} \displaystyle\int_x^{2x} \frac{dt}{t}$ ln 2

49. $\lim_{x\to 1} \dfrac{x^3 - 1}{4x^3 - x - 3}$ 3/11

50. $\lim_{x\to\infty} \dfrac{2x^2 + 3x}{x^3 + x + 1}$ 0

51. $\lim_{x\to 1} \dfrac{\int_1^x \cos t\, dt}{x^2 - 1}$ $(\cos 1)/2$

52. $\lim_{x\to 1} \dfrac{\int_1^x \frac{dt}{t}}{x^3 - 1}$ 1/3

Group Activity In Exercises 53 and 54, do the following.

(a) Writing to Learn Explain why l'Hôpital's Rule does not help you to find the limit.

(b) Use a graph to estimate the limit.

(c) Evaluate the limit analytically using the techniques of Chapter 2.

53. $\lim_{x\to\infty} \dfrac{\sqrt{9x + 1}}{\sqrt{x + 1}}$

54. $\lim_{x\to \pi/2} \dfrac{\sec x}{\tan x}$

55. *Continuous Extension* Find a value of c that makes the function

$$f(x) = \begin{cases} \dfrac{9x - 3\sin 3x}{5x^3}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

continuous at $x = 0$. Explain why your value of c works.

56. *Continuous Extension* Let $f(x) = |x|^x$, $x \neq 0$. Show that f has a removable discontinuity at $x = 0$ and extend the definition of f to $x = 0$ so that the extended function is continuous there.

57. *Interest Compounded Continuously*

(a) Show that $\lim_{k\to\infty} A_0\left(1 + \dfrac{r}{k}\right)^{kt} = A_0 e^{rt}$.

(b) Writing to Learn Explain how the limit in part (a) connects interest compounded k times per year with interest compounded continuously.

58. *L'Hôpital's Rule* Let

$$f(x) = \begin{cases} x + 2, & x \neq 0 \\ 0, & x = 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x + 1, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

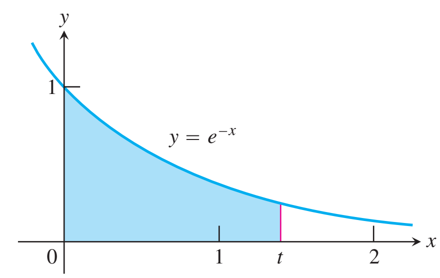
(a) Show that

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = 1 \quad \text{but} \quad \lim_{x\to 0} \frac{f(x)}{g(x)} = 2.$$

(b) Writing to Learn Explain why this does not contradict l'Hôpital's Rule.

59. *Solid of Revolution* Let $A(t)$ be the area of the region in the first quadrant enclosed by the coordinate axes, the curve $y = e^{-x}$, and the line $x = t > 0$ as shown in the figure. Let $V(t)$ be the volume of the solid generated by revolving the region about the x-axis. Find the following limits.

(a) $\lim_{t\to\infty} A(t)$ 1 **(b)** $\lim_{t\to\infty} \dfrac{V(t)}{A(t)}$ $\pi/2$ **(c)** $\lim_{t\to 0^+} \dfrac{V(t)}{A(t)}$ π



60. *L'Hôpital's Trap* Let $f(x) = \dfrac{1 - \cos x}{x + x^2}$.

(a) Use graphs or tables to estimate $\lim_{x\to 0} f(x)$. 0

(b) Find the error in the following incorrect application of l'Hôpital's Rule.

$$\lim_{x\to 0} \frac{1 - \cos x}{x + x^2} = \lim_{x\to 0} \frac{\sin x}{1 + 2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$$

L'Hôpital's Rule cannot be applied to $\dfrac{\sin x}{1 + 2x}$ because the denominator has limit 1.

61. *Exponential Functions* **(a)** Use the equation

$$a^x = e^{x \ln a}$$

to find the domain of

$$f(x) = \left(1 + \frac{1}{x}\right)^x. \quad (-\infty, -1)\cup(0, \infty)$$

(b) Find $\lim_{x\to -1^-} f(x)$. ∞

(c) Find $\lim_{x\to -\infty} f(x)$. e

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

62. True or False If $f(a) = g(a) = 0$ and $f'(a)$ and $g'(a)$ exist, then $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{f'(a)}{g'(a)}$. Justify your answer.

63. True or False $\lim_{x\to 0^+} x^x$ does not exist. Justify your answer. False. The limit is 1.

64. Multiple Choice Which of the following gives the value of C

$$\lim_{x\to 0} \frac{x}{\tan x}?$$

(A) -1 **(B)** 0 **(C)** 1 **(D)** π **(E)** Does not exist

65. Multiple Choice Which of the following gives the value of D

$$\lim_{x\to 1} \frac{1-\dfrac{1}{x}}{1-\dfrac{1}{x^2}}?$$

(A) Does not exist **(B)** 2 **(C)** 1 **(D)** 1/2 **(E)** 0

66. Multiple Choice Which of the following gives the value of B

$$\lim_{x\to\infty} \frac{\log_2 x}{\log_3 x}?$$

(A) 1 **(B)** $\dfrac{\ln 3}{\ln 2}$ **(C)** $\dfrac{\ln 2}{\ln 3}$ **(D)** $\ln\left(\dfrac{3}{2}\right)$ **(E)** $\ln\left(\dfrac{2}{3}\right)$

67. Multiple Choice Which of the following gives the value of E

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x}?$$

(A) 0 **(B)** 1 **(C)** e **(D)** e^2 **(E)** e^3

Explorations

68. Give an example of two differentiable functions f and g with $\lim_{x\to 3} f(x) = \lim_{x\to 3} g(x) = 0$ that satisfy the following.

(a) $\displaystyle\lim_{x\to 3}\frac{f(x)}{g(x)} = 7$ **(b)** $\displaystyle\lim_{x\to 3}\frac{f(x)}{g(x)} = 0$

(c) $\displaystyle\lim_{x\to 3}\frac{f(x)}{g(x)} = \infty$

Possible answers: **(a)** $f(x) = 7(x-3)$, $g(x) = x-3$
(b) $f(x) = (x-3)^2$, $g(x) = x-3$
(c) $f(x) = x-3$, $g(x) = (x-3)^3$

69. Give an example of two differentiable functions f and g with $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$ that satisfy the following.

(a) $\displaystyle\lim_{x\to\infty}\frac{f(x)}{g(x)} = 3$ **(b)** $\displaystyle\lim_{x\to\infty}\frac{f(x)}{g(x)} = 0$

(c) $\displaystyle\lim_{x\to\infty}\frac{f(x)}{g(x)} = \infty$

Possible answers: **(a)** $f(x) = 3x+1$, $g(x) = x$
(b) $f(x) = x+1$, $g(x) = x^2$
(c) $f(x) = x^2$, $g(x) = x+1$

Extending the Ideas

70. *Grapher Precision* Let $f(x) = \dfrac{1-\cos x^6}{x^{12}}$.

(a) Explain why some graphs of f may give false information about $\lim_{x\to 0} f(x)$. (*Hint:* Try the window $[-1, 1]$ by $[-0.5, 1]$.)

(b) Explain why tables may give false information about $\lim_{x\to 0} f(x)$. (*Hint:* Try tables with increments of 0.01.)

(c) Use l'Hôpital's Rule to find $\lim_{x\to 0} f(x)$.

(d) Writing to Learn This is an example of a function for which graphers do not have enough precision to give reliable information. Explain this statement in your own words.

71. *Cauchy's Mean Value Theorem* Suppose that functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and suppose also that $g' \neq 0$ throughout (a, b). Then there exists a number c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Find all values of c in (a, b) that satisfy this property for the following given functions and intervals.

(a) $f(x) = x^3+1$, $g(x) = x^2 - x$, $[a, b] = [-1, 1]$ $c = 1/3$

(b) $f(x) = \cos x$, $g(x) = \sin x$, $[a, b] = [0, \pi/2]$ $c = \pi/4$

72. *Why 0^∞ and $0^{-\infty}$ Are Not Indeterminate Forms* Assume that $f(x)$ is nonnegative in an open interval containing c and $\lim_{x\to c} f(x) = 0$.

(a) If $\displaystyle\lim_{x\to c} g(x) = \infty$, show that $\displaystyle\lim_{x\to c} f(x)^{g(x)} = 0$.

(b) If $\displaystyle\lim_{x\to c} g(x) = -\infty$, show that $\displaystyle\lim_{x\to c} f(x)^{g(x)} = \infty$.

Quick Quiz for AP* Preparation: Sections 8.1 and 8.2

You should solve the following problems without using a graphing calculator.

1. Multiple Choice Which of the following gives the value of

$$\lim_{x\to 0}\frac{(x+1)^{4/3}-(4/3)x-1}{x^2}? \quad \text{C}$$

(A) $-1/3$ **(B)** 0 **(C)** 2/9
(D) 4/9 **(E)** Does not exist

2. Multiple Choice Which of the following gives the value of $\lim_{x\to 0^+}(3x^{2x})$? D

(A) 0 **(B)** 1 **(C)** 2
(D) 3 **(E)** Does not exist

3. Multiple Choice Which of the following gives the value of

$$\lim_{x\to 2}\frac{\int_2^x \sin t\, dt}{x^2-4}? \quad \text{B}$$

(A) $-\dfrac{\sin 2}{4}$ **(B)** $\dfrac{\sin 2}{4}$ **(C)** $-\dfrac{\sin 2}{2}$

(D) $\dfrac{\sin 2}{2}$ **(E)** Does not exist

4. Free Response The second and fifth terms of a geometric sequence are -4 and $1/2$, respectively. Find

(a) the first term, 8 **(b)** the common ratio, $-1/2$

(c) an explicit rule for the nth term, and $a_n = (-1)^{n-1}(2^{4-n})$

(d) a recursive rule for the nth term. $a_n = (-1/2)a_{n-1}$

8.3 Relative Rates of Growth

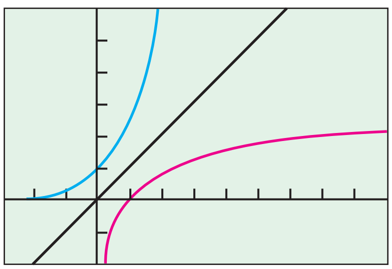
What you'll learn about

- Comparing Rates of Growth
- Using L'Hôpital's Rule to Compare Growth Rates
- Sequential versus Binary Search

. . . and why

Understanding growth rates as $x \to \infty$ is an important feature in understanding the behavior of functions.

$y = e^x$
$y = \ln x$
$y = x$



[–3, 9] by [–2, 6]

Figure 8.12 The graphs of $y = e^x$, $y = \ln x$, and $y = x$.

Grace Murray Hopper

(1906–1992)

Computer scientists use function comparisons like the ones in this section to measure the relative efficiencies of computer programs. The pioneering work of Rear Admiral Grace Murray Hopper in the field of computer technology led the navy, and the country, into the computer age. Hopper graduated from Yale in 1934 with a Ph.D. in Mathematics. During World War II she joined the navy and became director of a project that resulted in the development of COBOL, a computer language that enabled computers to "talk to one another." On September 6, 1997, the navy commissioned a multi-mission ship the "USS Hopper."

Comparing Rates of Growth

We restrict our attention to functions whose values eventually become and remain positive as $x \to \infty$.

The exponential function e^x grows so rapidly and the logarithm function $\ln x$ grows so slowly that they set standards by which we can judge the growth of other functions. The graphs (Figure 8.12) of e^x, $\ln x$, and x suggest how rapidly and slowly e^x and $\ln x$, respectively, grow in comparison to x.

In fact, all the functions a^x, $a > 1$, grow faster (eventually) than any power of x, even $x^{1{,}000{,}000}$ (Exercise 39), and hence faster (eventually) than any polynomial function.

To get a feeling for how rapidly the values of e^x grow with increasing x, think of graphing the function on a large blackboard, with the axes scaled in centimeters. At $x = 1$ cm, the graph is $e^1 \approx 3$ cm above the x-axis. At $x = 6$ cm, the graph is $e^6 \approx 403$ cm ≈ 4 m high. (It is about to go through the ceiling if it hasn't done so already.) At $x = 10$ cm, the graph is $e^{10} \approx 22{,}026$ cm ≈ 220 m high, higher than most buildings. At $x = 24$ cm, the graph is more than halfway to the moon, and at $x = 43$ cm from the origin, the graph is high enough to reach well past the sun's closest stellar neighbor, the red dwarf star Proxima Centauri:

$$\begin{aligned} e^{43} &\approx 4.7 \times 10^{18} \text{ cm} \\ &= 4.7 \times 10^{13} \text{ km} \\ &\approx 1.57 \times 10^{8} \text{ light-seconds} \\ &\approx 5.0 \text{ light-years.} \end{aligned}$$

Light travels about 300,000 km/sec in a vacuum.

The distance to Proxima Centauri is 4.2 light-years. Yet with $x = 43$ cm from the origin, the graph is still less than 2 feet to the right of the y-axis.

In contrast, the logarithm function $\ln x$ grows more slowly as $x \to \infty$ than any positive power of x, even $x^{1/1{,}000{,}000}$ (Exercise 41). Because $\ln x$ and e^x are inverse functions, the calculations above show that with axes scaled in centimeters, you have to go nearly 5 light-years out on the x-axis to find where the graph of $\ln x$ is even 43 cm high.

In fact, all the functions $\log_a x$, $a > 1$, grow slower (eventually) than any positive power of x.

These comparisons of exponential, polynomial, and logarithmic functions can be made precise by defining what it means for a function $f(x)$ to *grow faster* than another function $g(x)$ as $x \to \infty$.

DEFINITIONS Faster, Slower, Same-rate Growth as $x \to \infty$

Let $f(x)$ and $g(x)$ be positive for x sufficiently large.

1. f **grows faster** than g (and g **grows slower** than f) as $x \to \infty$ if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty, \quad \text{or, equivalently, if} \quad \lim_{x \to \infty} \frac{g(x)}{f(x)} = 0.$$

2. f and g **grow at the same rate** as $x \to \infty$ if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L \neq 0.$$

L finite and not zero

According to these definitions, $y = 2x$ does not grow faster than $y = x$ as $x \to \infty$. The two functions grow at the same rate because

$$\lim_{x \to \infty} \frac{2x}{x} = \lim_{x \to \infty} 2 = 2,$$

which is a finite nonzero limit. The reason for this apparent disregard of common sense is that we want "f grows faster than g" to mean that for large x-values, g is negligible in comparison to f.

If $L = 1$ in part 2 of the definition, then f and g are right end behavior models for each other (Section 2.2). If f grows faster than g, then

$$\lim_{x\to\infty} \frac{f(x) + g(x)}{f(x)} = \lim_{x\to\infty} \left(1 + \frac{g(x)}{f(x)}\right) = 1 + 0 = 1,$$

so f is a right end behavior model for $f + g$. Thus, for large x-values, g can be ignored in the sum $f + g$. This explains why, for large x-values, we can ignore the terms

$$g(x) = a_{n-1}x^{n-1} + \cdots + a_0$$

in

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0;$$

that is, why $a_n x^n$ is an end behavior model for

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0.$$

Using L'Hôpital's Rule to Compare Growth Rates

L'Hôpital's Rule can help us to compare rates of growth, as shown in Example 1.

EXAMPLE 1 Comparing e^x and x^2 as $x\to\infty$

Show that the function e^x grows faster than x^2 as $x\to\infty$.

SOLUTION

We need to show that $\lim_{x\to\infty} (e^x/x^2) = \infty$. Notice this limit is of indeterminate type ∞/∞, so we can apply l'Hôpital's Rule and take the derivative of the numerator and the derivative of the denominator. In fact, we have to apply l'Hôpital's Rule twice.

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{1} = \infty$$

Now try Exercise 1.

EXPLORATION 1 Comparing Rates of Growth as $x\to\infty$

1. Show that a^x, $a > 1$, grows faster than x^2 as $x\to\infty$.
2. Show that 3^x grows faster than 2^x as $x\to\infty$.
3. If $a > b > 1$, show that a^x grows faster than b^x as $x\to\infty$.

EXAMPLE 2 Comparing ln x with x and x^2 as $x\to\infty$

Show that ln x grows slower than **(a)** x and **(b)** x^2 as $x\to\infty$.

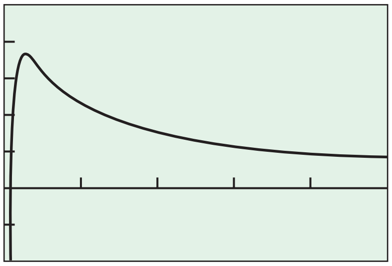
SOLUTION

(a) Solve Analytically

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} \qquad \text{l'Hôpital's Rule}$$

$$= \lim_{x\to\infty} \frac{1}{x} = 0$$

continued



[0, 50] by [–0.2, 0.5]

Figure 8.13 The x-axis is a horizontal asymptote of the function $f(x) = (\ln x)/x$. (Example 2)

Support Graphically Figure 8.13 suggests that the graph of the function $f(x) = (\ln x)/x$ drops dramatically toward the x-axis as x outstrips $\ln x$.

(b) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \left(\frac{\ln x}{x} \cdot \frac{1}{x}\right) = 0 \cdot 0 = 0$

Now try Exercise 5.

EXAMPLE 3 Comparing x with $x + \sin x$ as $x\to\infty$

Show that x grows at the same rate as $x + \sin x$ as $x\to\infty$.

SOLUTION

We need to show that $\lim_{x\to\infty} ((x + \sin x)/x)$ is finite and not 0. The limit can be computed directly.

$$\lim_{x\to\infty} \frac{x + \sin x}{x} = \lim_{x\to\infty} \left(1 + \frac{\sin x}{x}\right) = 1$$

Now try Exercise 9.

EXAMPLE 4 Comparing Logarithmic Functions as $x\to\infty$

Let a and b be numbers greater than 1. Show that $\log_a x$ and $\log_b x$ grow at the same rate as $x\to\infty$.

SOLUTION

$$\lim_{x\to\infty} \frac{\log_a x}{\log_b x} = \lim_{x\to\infty} \frac{\ln x/\ln a}{\ln x/\ln b} = \frac{\ln b}{\ln a}$$

The limiting value is finite and nonzero. ***Now try Exercise 13.***

Growing at the same rate is a *transitive relation.*

Transitivity of Growing Rates

If f grows at the same rate as g as $x\to\infty$ and g grows at the same rate as h as $x\to\infty$, then f grows at the same rate as h as $x\to\infty$.

The reason is that

$$\lim_{x\to\infty} \frac{f}{g} = L \quad \text{and} \quad \lim_{x\to\infty} \frac{g}{h} = M$$

together imply that

$$\lim_{x\to\infty} \frac{f}{h} = \lim_{x\to\infty} \left(\frac{f}{g} \cdot \frac{g}{h}\right) = LM.$$

If L and M are finite and nonzero, then so is LM.

EXAMPLE 5 Growing at the Same Rate as $x\to\infty$

Show that $f(x) = \sqrt{x^2 + 5}$ and $g(x) = (2\sqrt{x} - 1)^2$ grow at the same rate as $x\to\infty$.

SOLUTION

Solve Analytically We show that f and g grow at the same rate by showing that they both grow at the same rate as $h(x) = x$.

$$\lim_{x\to\infty} \frac{f(x)}{h(x)} = \lim_{x\to\infty} \frac{\sqrt{x^2 + 5}}{x} = \lim_{x\to\infty} \sqrt{1 + \frac{5}{x^2}} = 1$$

and

continued

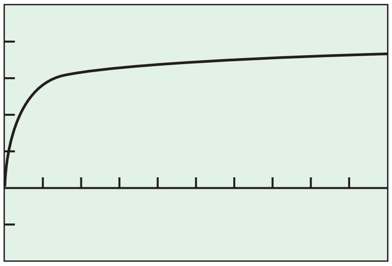
$$\lim_{x\to\infty}\frac{g(x)}{h(x)} = \lim_{x\to\infty}\frac{(2\sqrt{x}-1)^2}{x} = \lim_{x\to\infty}\left(\frac{2\sqrt{x}-1}{\sqrt{x}}\right)^2 = \lim_{x\to\infty}\left(2-\frac{1}{\sqrt{x}}\right)^2 = 4$$

Thus,

$$\lim_{x\to\infty}\frac{f}{g} = \lim_{x\to\infty}\left(\frac{f}{h}\cdot\frac{h}{g}\right) = 1\cdot\frac{1}{4} = \frac{1}{4},$$

and f and g grow at the same rate as $x\to\infty$.

Support Graphically The graph of $y = g/f$ in Figure 8.14 suggests that the quotient g/f is an increasing function with horizontal asymptote $y = 4$. This supports that f and g grow at the same rate. ***Now try Exercise 31.***



[0, 100] by [–2, 5]

Figure 8.14 The graph of g/f appears to have the line $y = 4$ as a horizontal asymptote. (Example 5)

Sequential versus Binary Search

Computer scientists sometimes measure the efficiency of an algorithm by counting the number of steps a computer must take to make the algorithm do something (Figure 8.15). (Your graphing calculator works according to algorithms programmed into it.) There can be significant differences in how efficiently algorithms perform, even if they are designed to accomplish the same task. Here is an example.

Webster's *Third New International Dictionary* lists about 26,000 words that begin with the letter *a*. One way to look up a word, or to learn if it is not there, is to read through the list one word at a time until you either find the word or determine that it is not there. This **sequential search** method makes no particular use of the words' alphabetical arrangement. You are sure to get an answer, but it might take about 26,000 steps.

Note

You would not use a sequential search method to find a word, but you might program a computer to search for a word using this technique.

Another way to find the word or to learn that it is not there is to go straight to the middle of the list (give or take a few words). If you do not find the word, then go to the middle of the half that would contain it and forget about the half that would not. (You know which half would contain it because you know the list is ordered alphabetically.) This **binary search** method eliminates roughly 13,000 words in this first step. If you do not find the word on the second try, then jump to the middle of the half that would contain it. Continue this way until you have found the word or divided the list in half so many times that there are no words left. How many times do you have to divide the list to find the word or learn that it is not there? At most 15, because

$$\frac{26{,}000}{2^{15}} < 1.$$

This certainly beats a possible 26,000 steps.

For a list of length n, a sequential search algorithm takes on the order of n steps to find a word or determine that it is not in the list.

EXAMPLE 6 Finding the Order of a Binary Search

For a list of length n, how many steps are required for a binary search?

SOLUTION

A binary search takes on the order of $\log_2 n$ steps. The reason is if $2^{m-1} < n \le 2^m$, then $m - 1 < \log_2 n \le m$, and the number of bisections required to narrow the list to one word will be at most m, the smallest integer greater than or equal to $\log_2 n$.
Now try Exercise 43.

Figure 8.15 Computer scientists look for the most efficient algorithms when they program searches.

On a list of length n, there is a big difference between a sequential search (order n) and a binary search (order $\log_2 n$) because n grows faster than $\log_2 n$ as $n\to\infty$. In fact,

$$\lim_{n\to\infty}\frac{n}{\log_2 n} = \lim_{n\to\infty}\frac{1}{n\ln 2} = \infty.$$

7. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty}\left(1 + \frac{\ln x}{x}\right) = 1 + 0 = 1$ **9. (a)** Local minimum at $(0, 1)$ Local maximum at $\approx(2, 1.541)$

Quick Review 8.3 *(For help, go to Sections 2.2 and 4.1.)*

In Exercises 1–4, evaluate the limit.

1. $\lim_{x\to\infty} \frac{\ln x}{e^x}$ 0

2. $\lim_{x\to\infty} \frac{e^x}{x^3}$ ∞

3. $\lim_{x\to-\infty} \frac{x^2}{e^{2x}}$ ∞

4. $\lim_{x\to\infty} \frac{x^2}{e^{2x}}$ 0

In Exercises 5 and 6, find an end behavior model (Section 2.2) for the function.

5. $f(x) = -3x^4 + 5x^3 - x + 1$ $-3x^4$

6. $f(x) = \frac{2x^3 - 3x + 1}{x + 2}$ $2x^2$

In Exercises 7 and 8, show that g is a right end behavior model for f.

7. $g(x) = x$, $f(x) = x + \ln x$

8. $g(x) = 2x$, $f(x) = \sqrt{4x^2 + 5x}$ $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \sqrt{1 + \frac{5}{4x}} = 1$

9. Let $f(x) = \frac{e^x + x^2}{e^x}$. Find the

(a) local extreme values of f and where they occur.

(b) intervals on which f is increasing. $[0, 2]$

(c) intervals on which f is decreasing. $(-\infty, 0]$ and $[2, \infty)$

10. Let $f(x) = \frac{x + \sin x}{x}$.

Find the absolute maximum value of f and where it occurs.

f doesn't have an absolute maximum value. The values are always less than 2 and the values get arbitrarily close to 2 near $x = 0$, but the function is undefined at $x = 0$.

13. $\lim_{x\to\infty} \frac{\log\sqrt{x}}{\ln x} = \frac{1}{2\ln 10}$

Section 8.3 Exercises

In Exercises 1–4, show that e^x grows faster than the given function.

1. $x^3 - 3x + 1$ $\lim_{x\to\infty} \frac{e^x}{x^3 - 3x + 1} = \infty$

2. x^{20} $\lim_{x\to\infty} \frac{e^x}{x^{20}} = \infty$

3. $e^{\cos x}$ $\lim_{x\to\infty} \frac{e^x}{e^{\cos x}} = \infty$

4. $(5/2)^x$ $\lim_{x\to\infty} \frac{e^x}{(5/2)^x} = \infty$

In Exercises 5–8, show that $\ln x$ grows slower than the given function.

5. $x - \ln x$ $\lim_{x\to\infty} \frac{\ln x}{x - \ln x} = 0$

6. $\sqrt{x}$ $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = 0$

7. $\sqrt[3]{x}$ $\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} = 0$

8. x^3 $\lim_{x\to\infty} \frac{\ln x}{x^3} = 0$

In Exercises 9–12, show that x^2 grows at the same rate as the given function.

9. $x^2 + 4x$ $\lim_{x\to\infty} \frac{x^2 + 4x}{x^2} = 1$

10. $\sqrt{x^4 + 5x}$ $\lim_{x\to\infty} \frac{\sqrt{x^4 + 5x}}{x^2} = 1$

11. $\sqrt[3]{x^6 + x^2}$ $\lim_{x\to\infty} \frac{\sqrt[3]{x^6 + x^2}}{x^2} = 1$

12. $x^2 + \sin x$ $\lim_{x\to\infty} \frac{x^2 + \sin x}{x^2} = 1$

In Exercises 13 and 14, show that the two functions grow at the same rate.

13. $\ln x$, $\log\sqrt{x}$

14. e^{x+1}, e^x $\lim_{x\to\infty} \frac{e^{x+1}}{e^x} = e$

In Exercises 15–20, determine whether the function grows faster than e^x, at the same rate as e^x, or slower than e^x as $x\to\infty$.

15. $\sqrt{1 + x^4}$ Slower

16. 4^x Faster

17. $x\ln x - x$ Slower

18. xe^x Faster

19. x^{1000} Slower

20. $(e^x + e^{-x})/2$ Same rate

In Exercises 21–24, determine whether the function grows faster than x^2, at the same rate as x^2, or slower than x^2 as $x\to\infty$.

21. $x^3 + 3$ Faster

22. $15x + 3$ Slower

23. $\ln x$ Slower

24. 2^x Faster

In Exercises 25–28, determine whether the function grows faster than $\ln x$, at the same rate as $\ln x$, or slower than $\ln x$ as $x\to\infty$.

25. $\log_2 x^2$ Same rate

26. $1/\sqrt{x}$ Slower

27. e^{-x} Slower

28. $5\ln x$ Same rate

In Exercises 29 and 30, order the functions from slowest-growing to fastest-growing as $x\to\infty$.

29. e^x, x^x, $(\ln x)^x$, $e^{x/2}$ $e^{x/2}$, e^x, $(\ln x)^x$, x^x

30. 2^x, x^2, $(\ln 2)^x$, e^x $(\ln 2)^x$, x^2, 2^x, e^x

In Exercises 31–34, show that the three functions grow at the same rate as $x\to\infty$.

31. $f_1(x) = \sqrt{x}$, $f_2(x) = \sqrt{10x + 1}$, $f_3(x) = \sqrt{x + 1}$

32. $f_1(x) = x^2$, $f_2(x) = \sqrt{x^4 + x}$, $f_3(x) = \sqrt{x^4 - x^3}$

33. $f_1(x) = 3^x$, $f_2(x) = \sqrt{9^x + 2^x}$, $f_3(x) = \sqrt{9^x - 4^x}$

34. $f_1(x) = x^3$, $f_2(x) = \frac{x^4 + 2x^2 - 1}{x + 1}$, $f_3(x) = \frac{2x^5 - 1}{x^2 + 1}$

In Exercises 35–38, only one of the following is true.

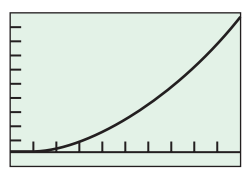
i. f grows faster than g.

ii. g grows faster than f.

iii. f and g grow at the same rate.

Use the given graph of f/g to determine which one is true.

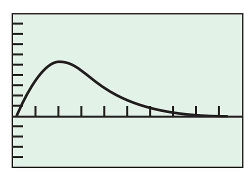
35.



[0, 100] by [–1000, 10000]

f grows faster than g

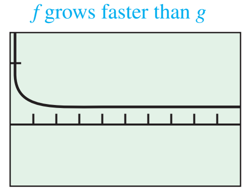
36.



[0, 10] by [–0.5, 1]

g grows faster than f

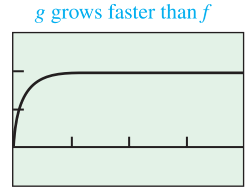
37.



[0, 100] by [–1, 1.5]

f and g grow at the same rate

38.



[0, 20] by [–1, 3]

f and g grow at the same rate

Group Activity In Exercises 39–41, do the following comparisons.

39. *Comparing Exponential and Power Functions*

(a) Writing to Learn Explain why e^x grows faster than x^n as $x\to\infty$ for any positive integer n, even $n = 1{,}000{,}000$. (*Hint:* What is the nth derivative of x^n?)

47. False. They grow at the same rate.

(b) Writing to Learn Explain why a^x, $a > 1$, grows faster than x^n as $x \to \infty$ for any positive integer n.

40. ***Comparing Exponential and Polynomial Functions***

(a) Writing to Learn Show that e^x grows faster than any polynomial

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \ a_n > 0,$$

as $x \to \infty$. Explain.

(b) Writing to Learn Show that a^x, $a > 1$, grows faster than any polynomial

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \ a_n > 0,$$

as $x \to \infty$. Explain.

41. ***Comparing Logarithm and Power Functions***

(a) Writing to Learn Show that $\ln x$ grows slower than $x^{1/n}$ as $x \to \infty$ for any positive integer n, even $n = 1{,}000{,}000$. Explain.

(b) Writing to Learn Show that for any number $a > 0$, $\ln x$ grows slower than x^a as $x \to \infty$. Explain.

42. ***Comparing Logarithm and Polynomial Functions*** Show that $\ln x$ grows slower than any nonconstant polynomial

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \ a_n > 0,$$

as $x \to \infty$.

43. ***Search Algorithms*** Suppose you have three different algorithms for solving the same problem and each algorithm provides for a number of steps that is of order of one of the functions listed here.

$$n \log_2 n, \quad n^{3/2}, \quad n(\log_2 n)^2$$

Which of the algorithms is likely the most efficient in the long run? Give reasons for your answer.

44. ***Sequential and Binary Search*** Suppose you are looking for an item in an ordered list one million items long. How many steps might it take to find the item with **(a)** a sequential search? **(b)** a binary search? **(a)** 1,000,000 **(b)** 20

45. ***Growing at the Same Rate*** Suppose that polynomials $p(x)$ and $q(x)$ grow at the same rate as $x \to \infty$. What can you conclude about

(a) $\displaystyle\lim_{x\to\infty} \frac{p(x)}{q(x)}$? **(b)** $\displaystyle\lim_{x\to-\infty} \frac{p(x)}{q(x)}$?

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

46. True or False A search of order $n \log_2 n$ is more efficient than a search of order $n^{3/2}$. Justify your answer. True, because $\displaystyle\lim_{n\to\infty} \frac{n \log_2 n}{n^{3/2}} = 0$.

47. True or False The function $f(x) = 100x^2 + 50x + 1$ grows faster than the function $x^2 + 1$ as $x \to \infty$. Justify your answer.

48. Multiple Choice Which of the following functions grows faster than $x^5 + x^2 + 1$ as $x \to \infty$? E

(A) $x^2 + 1$ **(B)** $x^3 + 2$ **(C)** $x^4 - x^2$ **(D)** x^5 **(E)** $x^6 + 1$

49. Multiple Choice Which of the following functions grows slower than $\log_{13} x$ as $x \to \infty$? A

(A) e^{-x} **(B)** $\log_2 x$ **(C)** $\ln x$ **(D)** $\log x$ **(E)** $x \ln x$

43. The one which is $O(n \log_2 n)$ is likely the most efficient, because of the three given functions, it grows the most slowly as $n \to \infty$.

52. **(a)** x^5 grows faster than x^2. **(b)** They grow at the same rate.

50. Multiple Choice Which of the following functions grows at the same rate as e^x as $x \to \infty$? C

(A) e^{2x} **(B)** e^{3x} **(C)** e^{x+2} **(D)** e^{-x} **(E)** e^{-x+1}

51. Multiple Choice Which of the following functions grows at the same rate as $\sqrt{x^8 + x^4}$ as $x \to \infty$? D

(A) x **(B)** x^2 **(C)** x^3 **(D)** x^4 **(E)** x^5

Explorations

52. Let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

and

$$g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$$

be any two polynomial functions with $a_n > 0$, $b_m > 0$.

(a) Compare the rates of growth of x^5 and x^2 as $x \to \infty$.

(b) Compare the rates of growth of $5x^3$ and $2x^3$ as $x \to \infty$.

(c) If x^m grows faster than x^n as $x \to \infty$, what can you conclude about m and n? $m > n$

(d) If x^m grows at the same rate as x^n as $x \to \infty$, what can you conclude about m and n? $m = n$

(e) If $g(x)$ grows faster than $f(x)$ as $x \to \infty$, what can you conclude about their degrees? $m > n$ (or, degree of $g >$ degree of f)

(f) If $g(x)$ grows at the same rate as $f(x)$ as $x \to \infty$, what can you conclude about their degrees? $m = n$ (or, degree of $g =$ degree of f)

Extending the Ideas

53. Suppose that the values of the functions $f(x)$ and $g(x)$ eventually become and remain negative as $x \to \infty$. We say that

i. f **decreases faster** than g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty.$$

ii. f and g **decrease at the same rate** as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \neq 0.$$

(a) Show that if f decreases faster than g as $x \to \infty$, then $|f|$ grows faster than $|g|$ as $x \to \infty$.

(b) Show that if f and g decrease at the same rate as $x \to \infty$, then $|f|$ and $|g|$ grow at the same rate as $x \to \infty$.

54. Suppose that the values of the functions $f(x)$ and $g(x)$ eventually become and remain positive as $x \to -\infty$. We say that

i. f **grows faster** than g as $x \to -\infty$ if

$$\lim_{x\to-\infty} \frac{f(x)}{g(x)} = \infty.$$

ii. f and g **grow at the same rate** as $x \to -\infty$ if

$$\lim_{x\to-\infty} \frac{f(x)}{g(x)} = L \neq 0.$$

(a) Show that if f grows faster than g as $x \to -\infty$, then $f(-x)$ grows faster than $g(-x)$ as $x \to \infty$.

(b) Show that if f and g grow at the same rate as $x \to -\infty$, then $f(-x)$ and $g(-x)$ grow at the same rate as $x \to \infty$.

45. **(a)** The limit will be the ratio of the leading coefficients of the polynomials. **(b)** The limit will be the same as in part (a).

8.4 Improper Integrals

What you'll learn about

- Infinite Limits of Integration
- Integrands with Infinite Discontinuities
- Test for Convergence and Divergence
- Applications

. . . and why

The techniques of this section allow us to extend integration techniques to cases where the interval of integration $[a, b]$ is not finite or where integrands are not continuous.

Infinite Limits of Integration

Consider the infinite region that lies under the curve $y = e^{-x/2}$ in the first quadrant (Figure 8.16a). You might think this region has infinite area, but we will see that it is finite. Here is how we assign a value to the area. First we find the area $A(b)$ of the portion of the region that is bounded on the right by $x = b$ (Figure 8.16b).

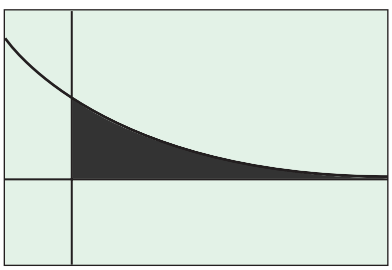
$$A(b) = \int_0^b e^{-x/2}\,dx = -2e^{-x/2}\Big]_0^b = -2e^{-b/2} + 2$$

Then we find the limit of $A(b)$ as $b \to \infty$.

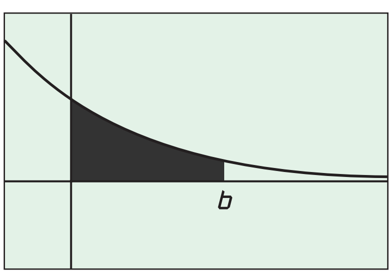
$$\lim_{b\to\infty} A(b) = \lim_{b\to\infty} (-2e^{-b/2} + 2) = 2$$

The area under the curve from 0 to ∞ is

$$\int_0^\infty e^{-x/2}\,dx = \lim_{b\to\infty} \int_0^b e^{-x/2}\,dx = 2.$$



(a)



(b)

Figure 8.16 (a) The area in the first quadrant under the curve $y = e^{-x/2}$ is (b) $\lim_{b\to\infty} \int_0^b e^{-x/2}\,dx.$

DEFINITION Improper Integrals with Infinite Integration Limits

Integrals with infinite limits of integration are **improper integrals.**

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

where c is any real number.

In parts 1 and 2, if the limit is finite the improper integral **converges** and the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges.** In part 3, the integral on the left-hand side of the equation **converges** if both improper integrals on the right-hand side converge, otherwise it **diverges** and has no value. It can be shown that the choice of c in part 3 is unimportant. We can evaluate or determine the convergence or divergence of $\int_{-\infty}^\infty f(x)\,dx$ with any convenient choice.

EXAMPLE 1 Writing Improper Integrals as Limits

Express the improper integral $\int_{-\infty}^\infty e^x\,dx$ in terms of limits of definite integrals and then evaluate the integral.

continued

SOLUTION

Choosing $c = 0$ in part 3 of the definition we can write the integral as

$$\int_{-\infty}^{\infty} e^x \, dx = \lim_{b \to -\infty} \int_b^0 e^x \, dx + \lim_{b \to \infty} \int_0^b e^x \, dx.$$

Next we evaluate the definite integrals and compute the corresponding limits.

$$\begin{aligned} \int_{-\infty}^{\infty} e^x \, dx &= \lim_{b \to -\infty} \int_b^0 e^x \, dx + \lim_{b \to \infty} \int_0^b e^x \, dx \\ &= \lim_{b \to -\infty} (1 - e^b) + \lim_{b \to \infty} (e^b - 1) \\ &= 1 + \infty \end{aligned}$$

The integral diverges because the second part diverges. ***Now try Exercise 3.***

EXAMPLE 2 Evaluating an Improper Integral on $[1, \infty)$

Does the improper integral $\displaystyle\int_1^{\infty} \frac{dx}{x}$ converge or diverge?

SOLUTION

$$\begin{aligned} \int_1^{\infty} \frac{dx}{x} &= \lim_{b \to \infty} \int_1^b \frac{dx}{x} && \text{Definition} \\ &= \lim_{b \to \infty} \ln x \Big]_1^b \\ &= \lim_{b \to \infty} (\ln b - \ln 1) = \infty \end{aligned}$$

Thus, the integral diverges. ***Now try Exercise 5.***

EXAMPLE 3 Using Partial Fractions with Improper Integrals

Evaluate $\displaystyle\int_0^{\infty} \frac{2\,dx}{x^2 + 4x + 3}$ or state that it diverges.

SOLUTION

By definition, $\displaystyle\int_0^{\infty} \frac{2\,dx}{x^2 + 4x + 3} = \lim_{b \to \infty} \int_0^b \frac{2\,dx}{x^2 + 4x + 3}$. We use partial fractions to integrate the definite integral. Set

$$\frac{2}{x^2 + 4x + 3} = \frac{A}{x + 1} + \frac{B}{x + 3}$$

and solve for A and B.

$$\begin{aligned} \frac{2}{x^2 + 4x + 3} &= \frac{A(x + 3)}{(x + 1)(x + 3)} + \frac{B(x + 1)}{(x + 3)(x + 1)} \\ &= \frac{(A + B)x + (3A + B)}{(x + 1)(x + 3)} \end{aligned}$$

continued

Thus, $A + B = 0$ and $3A + B = 2$. Solving, we find $A = 1$ and $B = -1$. Therefore,

$$\frac{2}{x^2 + 4x + 3} = \frac{1}{x+1} - \frac{1}{x+3}$$

and

$$\begin{aligned}
\int_0^b \frac{2\,dx}{x^2 + 4x + 3} &= \int_0^b \frac{dx}{x+1} - \int_0^b \frac{dx}{x+3} \\
&= \ln(x+1)\Big]_0^b - \ln(x+3)\Big]_0^b \\
&= \ln(b+1) - \ln(b+3) + \ln 3 \\
&= \ln\frac{b+1}{b+3} + \ln 3
\end{aligned}$$

So,

$$\lim_{b\to\infty}\left[\ln\left(\frac{b+1}{b+3}\right) + \ln 3\right] = \lim_{b\to\infty}\left[\ln\left(\frac{1+1/b}{1+3/b}\right) + \ln 3\right] = \ln 3.$$

Thus, $\displaystyle\int_0^\infty \frac{2\,dx}{x^2 + 4x + 3} = \ln 3.$ ***Now try Exercise 13.***

In Example 4 we use l'Hôpital's Rule to help evaluate the improper integral.

EXAMPLE 4 Using L'Hôpital's Rule with Improper Integrals

Evaluate $\int_1^\infty xe^{-x}\,dx$ or state that it diverges.

SOLUTION

By definition $\int_1^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_1^b xe^{-x}\,dx$. We use integration by parts to evaluate the definite integral. Let

$$\begin{aligned}
u &= x & dv &= e^{-x}\,dx \\
du &= dx & v &= -e^{-x}.
\end{aligned}$$

Then

$$\begin{aligned}
\int_1^b xe^{-x}\,dx &= \Big[-xe^{-x}\Big]_1^b + \int_1^b e^{-x}\,dx \\
&= \Big[-xe^{-x} - e^{-x}\Big]_1^b \\
&= \Big[-(x+1)e^{-x}\Big]_1^b \\
&= -(b+1)e^{-b} + 2e^{-1}.
\end{aligned}$$

So,

$$\begin{aligned}
\lim_{b\to\infty}\left[-(b+1)e^{-b} + 2e^{-1}\right] &= \lim_{b\to\infty}\frac{-(b+1)}{e^b} + \frac{2}{e} \\
&= \lim_{b\to\infty}\frac{-1}{e^b} + \frac{2}{e} \\
&= \frac{2}{e}.
\end{aligned}$$

Thus $\int_1^\infty xe^{-x}\,dx = 2/e.$ ***Now try Exercise 17.***

EXAMPLE 5 Evaluating an Integral on $(-\infty, \infty)$

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

SOLUTION

According to the definition (part 3) we can write

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}.$$

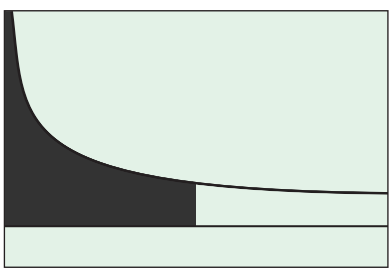
Next, we evaluate each improper integral on the right-hand side of the equation above.

$$\begin{aligned}\int_{-\infty}^{0} \frac{dx}{1+x^2} &= \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{1+x^2}\\ &= \lim_{a\to-\infty} \tan^{-1} x\Big]_a^0\\ &= \lim_{a\to-\infty} (\tan^{-1} 0 - \tan^{-1} a) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}\int_{0}^{\infty} \frac{dx}{1+x^2} &= \lim_{b\to\infty} \int_{0}^{b} \frac{dx}{1+x^2}\\ &= \lim_{b\to\infty} \tan^{-1} x\Big]_0^b\\ &= \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}\end{aligned}$$

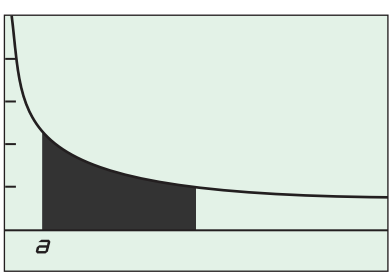
Thus,
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Now try Exercise 21.



[0, 2] by [–1, 5]

(a)



[0, 2] by [–1, 5]

(b)

Figure 8.17 (a) The area under the curve $y = 1/\sqrt{x}$ from $x = 0$ to $x = 1$ is (b)

$$\lim_{a\to0^+} \int_a^1 (1/\sqrt{x})\,dx.$$

Integrands with Infinite Discontinuities

Another type of improper integral arises when the integrand has a vertical asymptote—an infinite discontinuity—at a limit of integration or at some point between the limits of integration.

Consider the infinite region in the first quadrant that lies under the curve $y = 1/\sqrt{x}$ from $x = 0$ to $x = 1$ (Figure 8.17a). First we find the area of the portion from a to 1 (Figure 8.17b).

$$\int_a^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big]_a^1 = 2 - 2\sqrt{a}$$

Then, we find the limit of this area as $a \to 0^+$.

$$\lim_{a\to0^+} \int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a\to0^+} (2 - 2\sqrt{a}) = 2$$

The area under the curve from 0 to 1 is

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a\to0^+} \int_a^1 \frac{dx}{\sqrt{x}} = 2.$$

DEFINITION Improper Integrals with Infinite Discontinuities

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals.**

1. If $f(x)$ is continuous on $(a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In parts 1 and 2, if the limit is finite the improper integral **converges** and the limit is the **value** of the improper integral. If the limit fails to exist the improper integral **diverges.** In part 3, the integral on the left-hand side of the equation **converges** if both integrals on the right-hand side have values, otherwise it **diverges.**

EXPLORATION 1 Investigation $\int_0^1 \frac{dx}{x^p}$

1. Explain why these integrals are improper if $p > 0$.
2. Show that the integral diverges if $p = 1$.
3. Show that the integral diverges if $p > 1$.
4. Show that the integral converges if $0 < p < 1$.

EXAMPLE 6 Infinite Discontinuity at an Interior Point

Evaluate $\displaystyle\int_0^3 \frac{dx}{(x-1)^{2/3}}$.

SOLUTION

The integrand has a vertical asymptote at $x = 1$ and is continuous on $[0, 1)$ and $(1, 3]$. Thus, by part 3 of the definition above

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}.$$

Next, we evaluate each improper integral on the right-hand side of this equation.

$$\begin{aligned}\int_0^1 \frac{dx}{(x-1)^{2/3}} &= \lim_{c \to 1^-} \int_0^c \frac{dx}{(x-1)^{2/3}} \\ &= \lim_{c \to 1^-} 3(x-1)^{1/3}\Big]_0^c \\ &= \lim_{c \to 1^-} \left[3(c-1)^{1/3} + 3\right] = 3\end{aligned}$$

continued

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{c\to1^+}\int_c^3 \frac{dx}{(x-1)^{2/3}}$$
$$= \lim_{c\to1^+} 3(x-1)^{1/3}\Big]_c^3$$
$$= \lim_{c\to1^+} [3(3-1)^{1/3} - 3(c-1)^{1/3}] = 3\sqrt[3]{2}$$

We conclude that

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\sqrt[3]{2}.$$

Now try Exercise 25.

EXAMPLE 7 Infinite Discontinuity at an Endpoint

Evaluate $\displaystyle\int_1^2 \frac{dx}{x-2}$.

SOLUTION

The integrand has an infinite discontinuity at $x = 2$ and is continuous on $[1, 2)$. Thus,

$$\int_1^2 \frac{dx}{x-2} = \lim_{c\to2^-}\int_1^c \frac{dx}{x-2}$$
$$= \lim_{c\to2^-} \ln|x-2|\Big]_1^c$$
$$= \lim_{c\to2^-} (\ln|c-2| - \ln|-1|) = -\infty.$$

The original integral diverges and has no value. *Now try Exercise 29.*

Test for Convergence and Divergence

When we cannot evaluate an improper integral directly (often the case in practice) we first try to determine whether it converges or diverges. If the integral diverges, that's the end of the story. If it converges, we can then use numerical methods to approximate its value. In such cases the following theorem is useful.

THEOREM 6 Comparison Test

Let f and g be continuous on $[a, \infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then

1. $\displaystyle\int_a^\infty f(x)\,dx$ converges if $\displaystyle\int_a^\infty g(x)\,dx$ converges.

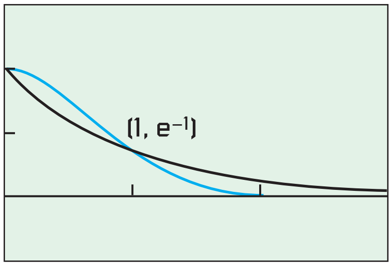
2. $\displaystyle\int_a^\infty g(x)\,dx$ diverges if $\displaystyle\int_a^\infty f(x)\,dx$ diverges.

EXAMPLE 8 Investigating Convergence

Does the integral $\int_1^\infty e^{-x^2}\,dx$ converge?

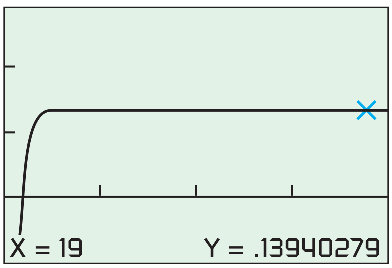
SOLUTION

Solve Analytically By definition, $\displaystyle\int_1^\infty e^{-x^2}\,dx = \lim_{b\to\infty}\int_1^b e^{-x^2}\,dx.$



[0, 3] by [–0.5, 1.5]

Figure 8.18 The graph of $y = e^{-x^2}$ lies below the graph of $y = e^{-x}$ for $x > 1$. (Example 8)



[0, 20] by [–0.1, 0.3]

Figure 8.19 The graph of NINT $(e^{-x^2}, x, 1, x)$. (Example 8)

We cannot evaluate the latter integral directly because there is no simple formula for the antiderivative of e^{-x^2}. We must therefore determine its convergence or divergence some other way. Because $e^{-x^2} > 0$ for all x, $\int_1^b e^{-x^2}\,dx$ is an increasing function of b. Therefore, as $b \to \infty$, the integral either becomes infinite as $b \to \infty$ or it is bounded from above and is forced to converge (have a finite limit).

The two curves $y = e^{-x^2}$ and $y = e^{-x}$ intersect at $(1, e^{-1})$, and $0 < e^{-x^2} \le e^{-x}$ for $x \ge 1$ (Figure 8.18). Thus, for any $b > 1$,

$$0 < \int_1^b e^{-x^2}\,dx \le \int_1^b e^{-x}\,dx = -e^{-b} + e^{-1} < e^{-1} \approx 0.368.$$

Rounded up to be safe

As an increasing function of b bounded above by 0.368, the integral $\int_1^\infty e^{-x^2}\,dx$ must converge. This does not tell us much about the value of the improper integral, however, except that it is positive and less than 0.368.

Support Graphically The graph of NINT $(e^{-x^2}, x, 1, x)$ is shown in Figure 8.19. The value of the integral rises rapidly as x first moves away from 1 but changes little past $x = 3$. Values sampled along the curve suggest a limit of about 0.13940 as $x \to \infty$. (Exercise 57 shows how to confirm the accuracy of this estimate.)

Now try Exercise 31.

Applications

EXAMPLE 9 Finding Circumference

Use the arc length formula (Section 7.4) to show that the circumference of the circle $x^2 + y^2 = 4$ is 4π.

SOLUTION

One fourth of this circle is given by $y = \sqrt{4 - x^2}$, $0 \le x \le 2$. Its arc length is

$$L = \int_0^2 \sqrt{1 + (y')^2}\,dx, \quad \text{where} \quad y' = -\frac{x}{\sqrt{4 - x^2}}.$$

The integral is improper because y' is not defined at $x = 2$. We evaluate it as a limit.

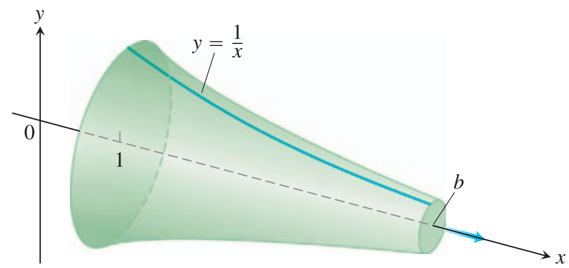
$$\begin{aligned}
L = \int_0^2 \sqrt{1 + (y')^2}\,dx &= \int_0^2 \sqrt{1 + \frac{x^2}{4 - x^2}}\,dx \\
&= \int_0^2 \sqrt{\frac{4}{4 - x^2}}\,dx \\
&= \lim_{b \to 2^-} \int_0^b \sqrt{\frac{4}{4 - x^2}}\,dx \\
&= \lim_{b \to 2^-} \int_0^b \sqrt{\frac{1}{1 - (x/2)^2}}\,dx \\
&= \lim_{b \to 2^-} 2\sin^{-1}\frac{x}{2}\Big]_0^b \\
&= \lim_{b \to 2^-} 2\left[\sin^{-1}\frac{b}{2} - 0\right] = \pi
\end{aligned}$$

The circumference of the quarter circle is π; the circumference of the circle is 4π.

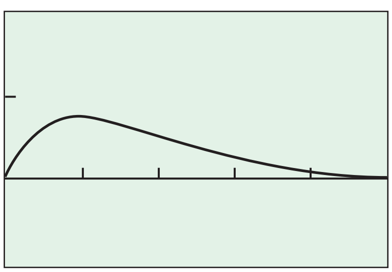
Now try Exercise 47.

EXPLORATION 2 Gabriel's Horn

Consider the region R in the first quadrant bounded above by $y = 1/x$ and on the left by $x = 1$. The region is revolved about the x-axis to form an infinite solid called Gabriel's Horn, which is shown in the figure.



1. Explain how Example 1 shows that the region R has infinite area.
2. Find the volume of the solid.
3. Find the area of the shadow that would be cast by Gabriel's Horn.
4. Why is Gabriel's Horn sometimes described as a solid that has finite volume but casts an infinite shadow?



[0, 5] by [–0.5, 1]

Figure 8.20 The graph of $y = xe^{-x}$. (Example 10)

EXAMPLE 10 Finding the Volume of an Infinite Solid

Find the volume of the solid obtained by revolving the curve $y = xe^{-x}$, $0 \le x < \infty$ about the x-axis.

SOLUTION

Figure 8.20 shows a portion of the region to be revolved about the x-axis. The area of a typical cross section of the solid is

$$\pi(\text{radius})^2 = \pi y^2 = \pi x^2 e^{-2x}.$$

The volume of the solid is

$$V = \pi \int_0^\infty x^2 e^{-2x}\,dx = \pi \lim_{b\to\infty} \int_0^b x^2 e^{-2x}\,dx.$$

Integrating by parts twice we obtain the following.

$$\int x^2 e^{-2x}\,dx = -\frac{x^2}{2}e^{-2x} + \int xe^{-2x}\,dx \qquad u = x^2,\ dv = e^{-2x}\,dx;\ du = 2x\,dx,\ v = -\tfrac{1}{2}e^{-2x}$$

$$= -\frac{x^2}{2}e^{-2x} - \frac{x}{2}e^{-2x} + \frac{1}{2}\int e^{-2x}\,dx \qquad u = x,\ dv = e^{-2x}\,dx;\ du = dx,\ v = -\tfrac{1}{2}e^{-2x}$$

$$= -\frac{x^2}{2}e^{-2x} - \frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + C$$

$$= -\frac{2x^2 + 2x + 1}{4e^{2x}} + C$$

continued

Thus,

$$V = \pi \lim_{b\to\infty}\left[-\frac{2x^2+2x+1}{4e^{2x}}\right]_0^b$$

$$= \pi \lim_{b\to\infty}\left[-\frac{2b^2+2b+1}{4e^{2b}}+\frac{1}{4}\right] = \frac{\pi}{4},$$

and the volume of the solid is $\pi/4$.

Now try Exercise 55.

Quick Review 8.4 *(For help, go to Sections 1.2, 5.3, and 8.2.)*

In Exercises 1–4, evaluate the integral.

1. $\displaystyle\int_0^3 \frac{dx}{x+3}$ $\ln 2$

2. $\displaystyle\int_{-1}^1 \frac{x\,dx}{x^2+1}$ 0

3. $\displaystyle\int \frac{dx}{x^2+4}$ $\frac{1}{2}\tan^{-1}\frac{x}{2}+C$

4. $\displaystyle\int \frac{dx}{x^4}$ $-\frac{1}{3}x^{-3}+C$

In Exercises 5 and 6, find the domain of the function.

5. $g(x) = \dfrac{1}{\sqrt{9-x^2}}$ $(-3, 3)$

6. $h(x) = \dfrac{1}{\sqrt{x-1}}$ $(1, \infty)$

In Exercises 7 and 8, confirm the inequality.

7. $\left|\dfrac{\cos x}{x^2}\right| \le \dfrac{1}{x^2}$, $-\infty < x < \infty$ Because $-1 \le \cos x \le 1$ for all x

8. $\dfrac{1}{\sqrt{x^2-1}} \ge \dfrac{1}{x}$, $x > 1$ Because $\sqrt{x^2-1} < \sqrt{x^2} = x$ for $x > 1$

In Exercises 9 and 10, show that the functions f and g grow at the same rate as $x\to\infty$.

9. $f(x) = 4e^x - 5$, $g(x) = 3e^x + 7$ $\displaystyle\lim_{x\to\infty}\frac{4e^x-5}{3e^x+7} = \frac{4}{3}$

10. $f(x) = \sqrt{2x-1}$, $g(x) = \sqrt{x+3}$ $\displaystyle\lim_{x\to\infty}\frac{\sqrt{2x-1}}{\sqrt{x+3}} = \sqrt{2}$

Section 8.4 Exercises

In Exercises 1–4, **(a)** express the improper integral as a limit of definite integrals, and **(b)** evaluate the integral.

1. $\displaystyle\int_0^\infty \frac{2x}{x^2+1}\,dx$ **(a)** $\displaystyle\lim_{b\to\infty}\int_0^b \frac{2x}{x^2+1}\,dx$ **(b)** ∞, diverges

2. $\displaystyle\int_1^\infty \frac{dx}{x^{1/3}}$ **(a)** $\displaystyle\lim_{b\to\infty}\int_1^b \frac{dx}{x^{1/3}}$ **(b)** ∞, diverges

3. $\displaystyle\int_{-\infty}^\infty \frac{2x}{(x^2+1)^2}\,dx$

4. $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}}$ **(a)** $\displaystyle\lim_{b\to\infty}\int_1^b \frac{dx}{\sqrt{x}}$ **(b)** ∞, diverges

In Exercises 5–24, evaluate the improper integral or state that it diverges.

5. $\displaystyle\int_1^\infty \frac{dx}{x^4}$ $1/3$

6. $\displaystyle\int_1^\infty \frac{2dx}{x^3}$ 1

7. $\displaystyle\int_1^\infty \frac{dx}{\sqrt[3]{x}}$ diverges

8. $\displaystyle\int_1^\infty \frac{dx}{\sqrt[4]{x}}$ diverges

9. $\displaystyle\int_{-\infty}^{-1} \frac{dx}{x^2}$ 1

10. $\displaystyle\int_{-\infty}^{0} \frac{dx}{(x-2)^3}$ $-1/8$

11. $\displaystyle\int_{-\infty}^{-2} \frac{2dx}{x^2-1}$ $\ln(3)$

12. $\displaystyle\int_2^\infty \frac{3dx}{x^2-x}$ $3\ln(2)$

13. $\displaystyle\int_{-1}^\infty \frac{dx}{x^2+5x+6}$ $\ln(2)$

14. $\displaystyle\int_{-\infty}^{0} \frac{2dx}{x^2-4x+3}$ $\ln(3)$

3. (a) $\displaystyle\lim_{b\to-\infty}\int_b^0 \frac{2x}{(x^2+1)^2}\,dx + \lim_{b\to\infty}\int_0^b \frac{2x}{(x^2+1)^2}\,dx$
(b) 0, converges

15. $\displaystyle\int_1^\infty \frac{5x+6}{x^2+2x}\,dx$ diverges

16. $\displaystyle\int_{-2}^{-\infty} \frac{2dx}{x^2-2x}$ $-\ln(2)$

17. $\displaystyle\int_1^\infty xe^{-2x}\,dx$ $(3/4)\,e^{-2}$

18. $\displaystyle\int_{-\infty}^0 x^2e^x\,dx$ 2

19. $\displaystyle\int_1^\infty x\ln(x)\,dx$ diverges

20. $\displaystyle\int_0^\infty (x+1)e^{-x}\,dx$ 2

21. $\displaystyle\int_{-\infty}^\infty e^{-|x|}\,dx$ 2

22. $\displaystyle\int_{-\infty}^\infty 2xe^{-x^2}\,dx$ 0

23. $\displaystyle\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}}$ $\pi/2$

24. $\displaystyle\int_{-\infty}^\infty e^{2x}\,dx$ diverges

In Exercises 25–30, **(a)** state why the integral is improper. Then **(b)** evaluate the integral or state that it diverges.

25. $\displaystyle\int_0^2 \frac{dx}{1-x^2}$ See page 468.

26. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ See page 468.

27. $\displaystyle\int_0^1 \frac{x+1}{\sqrt{x^2+2x}}\,dx$ See page 468.

28. $\displaystyle\int_0^4 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$ See page 468.

29. $\displaystyle\int_0^1 x\ln(x)dx$ See page 468.

30. $\displaystyle\int_{-1}^4 \frac{dx}{\sqrt{|x|}}$ See page 468.

In Exercises 31–34, use the Comparison Test to determine whether the integral converges or diverges.

31. $\displaystyle\int_1^{\infty} \frac{dx}{1+e^x}$ See page 469.

32. $\displaystyle\int_1^{\infty} \frac{dx}{x^3+1}$ See page 469.

33. $\displaystyle\int_{\pi}^{\infty} \frac{2+\cos x}{x}\,dx$ See page 469.

34. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^4+1}}$ See page 469.

In Exercises 35–42, evaluate the integral or state that it diverges.

35. $\displaystyle\int_0^{\ln 2} y^{-2}e^{1/y}\,dy$ diverges

36. $\displaystyle\int_0^{4} \frac{dr}{\sqrt{4-r}}$ 4

37. $\displaystyle\int_0^{\infty} \frac{ds}{(1+s)\sqrt{s}}$ π

38. $\displaystyle\int_1^{2} \frac{du}{u\sqrt{u^2-1}}$ $\pi/3$

39. $\displaystyle\int_0^{\infty} \frac{16\tan^{-1} v}{1+v^2}\,dv$ $2\pi^2$

40. $\displaystyle\int_{-\infty}^{0} \theta e^{\theta}\,d\theta$ -1

41. $\displaystyle\int_0^{2} \frac{dt}{1-t}$ diverges

42. $\displaystyle\int_{-1}^{1} \ln(|w|)dw$ -2

In Exercises 43 and 44, find the area of the region in the first quadrant that lies under the given curve.

43. $y = \dfrac{\ln x}{x^2}$ 1

44. $y = \dfrac{\ln x}{x}$ ∞

45. Group Activity

(a) Show that if f is an even function and the necessary integrals exist, then

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\int_0^{\infty} f(x)\,dx.$$

(b) Show that if f is odd and the necessary integrals exist, then

$$\int_{-\infty}^{\infty} f(x)\,dx = 0.$$

46. Writing to Learn

(a) Show that the integral $\displaystyle\int_0^{\infty} \frac{2x\,dx}{x^2+1}$ diverges.

(b) Explain why we can conclude from part (a) that

$$\int_{-\infty}^{\infty} \frac{2x\,dx}{x^2+1} \quad \text{diverges.}$$

(c) Show that $\displaystyle\lim_{b\to\infty} \int_{-b}^{b} \frac{2x\,dx}{x^2+1} = 0.$

(d) Explain why the result in part (c) does not contradict part (b).

47. ***Finding Perimeter*** Find the perimeter of the 4-sided figure $x^{2/3} + y^{2/3} = 1$. 6

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

In Exercises 48 and 49, let f and g be continuous on $[a, \infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$.

48. True or False If $\int_a^{\infty} f(x)dx$ converges then $\int_a^{\infty} g(x)dx$ converges. Justify your answer. False. See Theorem 6.

49. True or False If $\int_a^{\infty} g(x)dx$ converges then $\int_a^{\infty} f(x)dx$ converges. Justify your answer. True. See Theorem 6.

50. Multiple Choice Which of the following gives the value of the integral $\displaystyle\int_1^{\infty} \frac{dx}{x^{1.01}}$? C

(A) 1 **(B)** 10 **(C)** 100 **(D)** 1000 **(E)** diverges

51. Multiple Choice Which of the following gives the value of the integral $\displaystyle\int_0^{1} \frac{dx}{x^{0.5}}$? B

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** diverges

52. Multiple Choice Which of the following gives the value of the integral $\displaystyle\int_0^{1} \frac{dx}{x-1}$? E

(A) -1 **(B)** $-1/2$ **(C)** 0 **(D)** 1 **(E)** diverges

53. Multiple Choice Which of the following gives the value of the area under the curve $y = 1/(x^2+1)$ in the first quadrant? C

(A) $\pi/4$ **(B)** 1 **(C)** $\pi/2$ **(D)** π **(E)** diverges

Explorations

54. ***The Integral*** $\displaystyle\int_1^{\infty} \frac{dx}{x^p}$.

(a) Evaluate the integral for $p = 0.5$. ∞, or diverges

(b) Evaluate the integral for $p = 1$. ∞, or diverges

(c) Evaluate the integral for $p = 1.5$. 2

(d) Show that $\displaystyle\int_1^{\infty} \frac{dx}{x^p} = \lim_{b\to\infty}\left[\frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right)\right]$

(e) Use part (d) to show that $\displaystyle\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1}, & p > 1 \\ \infty, & p < 1. \end{cases}$

(f) For what values of p does the integral converge? diverge?
converges for $p > 1$, diverges for $p \le 1$

25. (a) The integral has an infinite discontinuity at the interior point $x = 1$.
(b) diverges

26. (a) The integral has an infinite discontinuity at the endpoint $x = 1$.
(b) $\pi/2$

27. (a) The integral has an infinite discontinuity at the endpoint $x = 0$.
(b) $\sqrt{3}$

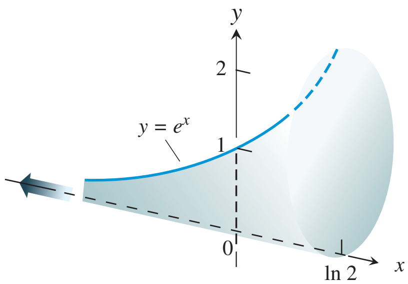
28. (a) The integral has an infinite discontinuity at the endpoint $x = 0$.
(b) $2 - 2e^{-2}$

29. (a) The integral has an infinite discontinuity at the endpoint $x = 0$.
(b) $-1/4$

30. (a) The integral has an infinite discontinuity at the interior point $x = 0$.
(b) 6

55. (b) $V = \int_{-\infty}^{\ln 2} A(x)dx = \int_{-\infty}^{\ln 2} (\pi/4)\, e^{2x}\, dx$

55. Each cross section of the solid infinite horn shown in the figure cut by a plane perpendicular to the x-axis for $-\infty < x \le \ln 2$ is a circular disc with one diameter reaching from the x-axis to the curve $y = e^x$.



(a) Find the area of a typical cross section. $A(x) = (\pi/4)\, e^{2x}$

(b) Express the volume of the horn as an improper integral.

(c) Find the volume of the horn. $\pi/2$

56. *Normal Probability Distribution Function* In Section 7.5, we encountered the bell-shaped normal distribution curve that is the graph of

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2},$$

the normal probability density function with mean μ and standard deviation σ. The number μ tells where the distribution is centered, and σ measures the "scatter" around the mean.

From the theory of probability, it is known that

$$\int_{-\infty}^{\infty} f(x)\, dx = 1.$$

In what follows, let $\mu = 0$ and $\sigma = 1$.

(a) Draw the graph of f. Find the intervals on which f is increasing, the intervals on which f is decreasing, and any local extreme values and where they occur.

(b) Evaluate $\int_{-n}^{n} f(x)\, dx$ for $n = 1, 2, 3$.

(c) Give a convincing argument that $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

(*Hint:* Show that $0 < f(x) < e^{-x/2}$ for $x > 1$, and for $b > 1$, $\int_{b}^{\infty} e^{-x/2}\, dx \to 0$ as $b \to \infty$.)

57. *Approximating the Value of $\int_1^\infty e^{-x^2}\, dx$*

(a) Show that $\int_{6}^{\infty} e^{-x^2}\, dx \le \int_{6}^{\infty} e^{-6x}\, dx < 4 \times 10^{-17}$.

(b) Writing to Learn Explain why

$$\int_{1}^{\infty} e^{-x^2}\, dx \approx \int_{1}^{6} e^{-x^2}\, dx$$

with error of at most 4×10^{-17}.

(c) Use the approximation in part (b) to estimate the value of $\int_1^\infty e^{-x^2}\, dx$. Compare this estimate with the value displayed in Figure 8.19.

(d) Writing to Learn Explain why

$$\int_{0}^{\infty} e^{-x^2}\, dx \approx \int_{0}^{3} e^{-x^2}\, dx$$

with error of at most 0.000042.

Extending the Ideas

58. Use properties of integrals to give a convincing argument that Theorem 6 is true.

59. Consider the integral

$$f(n+1) = \int_{0}^{\infty} x^n e^{-x}\, dx$$

where $n \ge 0$.

(a) Show that $\int_0^\infty x^n e^{-x}\, dx$ converges for $n = 0, 1, 2$.

(b) Use integration by parts to show that $f(n+1) = nf(n)$.

(c) Give a convincing argument that $\int_0^\infty x^n e^{-x}\, dx$ converges for all integers $n \ge 0$.

60. Let $f(x) = \int_{0}^{x} \frac{\sin t}{t}\, dt.$

(a) Use graphs and tables to investigate the values of $f(x)$ as $x \to \infty$.

(b) Does the integral $\int_0^\infty (\sin x)/x\, dx$ converge? Give a convincing argument.

61. (a) Show that we get the same value for the improper integral in Example 5 if we express

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{1} \frac{dx}{1+x^2} + \int_{1}^{\infty} \frac{dx}{1+x^2},$$

and then evaluate these two integrals.

(b) Show that it doesn't matter what we choose for c in (Improper Integrals with Infinite Integration Limits, part 3)

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{c} f(x)\, dx + \int_{c}^{\infty} f(x)\, dx.$$

31. $0 \le \frac{1}{1+e^x} \le \frac{1}{e^x}$ on $[1, \infty)$, converges because $\int_1^\infty \frac{1}{e^x}\, dx$ converges

32. $0 \le \frac{1}{x^3+1} \le \frac{1}{x^3}$ on $[1, \infty)$, converges because $\int_1^\infty \frac{1}{x^3}\, dx$ converges

33. $0 \le \frac{1}{x} \le \frac{2+\cos x}{x}$ on $[\pi, \infty)$, diverges because $\int_\pi^\infty \frac{1}{x}\, dx$ diverges

34. $\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^4+1}} = 2\int_0^\infty \frac{dx}{\sqrt{x^4+1}} = 2\int_0^1 \frac{dx}{\sqrt{x^4+1}} + 2\int_1^\infty \frac{dx}{\sqrt{x^4+1}}$ and $0 \le \frac{1}{\sqrt{x^4+1}} \le \frac{1}{x^2}$ on $[1, \infty)$, converges because $\int_1^\infty \frac{1}{x^2}\, dx$ converges

Quick Quiz for AP* Preparation: Sections 8.3 and 8.4

You may use a graphing calculator to solve the following problems.

1. **Multiple Choice** Which of the following functions grows faster than x^2 as $x \to \infty$? E
   (A) e^{-x} (B) $\ln(x)$ (C) $7x + 10$ (D) $2x^2 - 3x$ (E) $0.1x^3$
2. **Multiple Choice** Find all the values of p for which the integral converges $\int_1^{\infty} \frac{dx}{x^{p+1}}$. C
   (A) $p < -1$ (B) $p < 0$ (C) $p > 0$
   (D) $p > 1$ (E) diverges for all p
3. **Multiple Choice** Find all the values of p for which the integral converges $\int_0^1 \frac{dx}{x^{p+1}}$. B
   (A) $p < -1$ (B) $p < 0$ (C) $p > 0$
   (D) $p > 1$ (E) diverges for all p
4. **Free Response** Consider the region R in the first quadrant under the curve $y = \frac{2\ln(x)}{x^2}$.
   (a) Write the area of R as an improper integral.
   (b) Express the integral in part (a) as a limit of a definite integral.
   (c) Find the area of R.

Chapter 8 Key Terms

Absolute Value Theorem for Sequences (p. 440)
arithmetic sequence (p. 436)
binary search (p. 456)
common difference (p. 436)
common ratio (p. 437)
comparison test (p. 464)
constant multiple rule for limits (p. 439)
convergence of improper integral (p. 459)
convergent sequence (p. 439)
difference rule for limits (p. 439)
divergence of improper integral (p. 459)
divergent sequence (p. 439)
explicitly defined sequence (p. 435)
finite sequence (p. 435)
geometric sequence (p. 437)
grows at the same rate (p. 453)
grows faster (p. 453)
grows slower (p. 453)
improper integral (pp. 459)
indeterminate form (p. 444)
infinite sequence (p. 435)
l'Hôpital's Rule, first form (p. 444)
l'Hôpital's Rule, stronger form (p. 445)
limit of a sequence (p. 439)
nth term of a sequence (p. 435)
product rule for limits (p. 439)
quotient rule for limits (p. 439)
recursively defined sequence (p. 435)
Sandwich Theorem for Sequences (p. 440)
sequence (p. 435)
sequential search (p. 456)
sum rule for limits (p. 439)
terms of sequence (p. 435)
transitivity of growing rates (p. 455)
value of improper integral (p. 459)

Chapter 8 Review Exercises

The collection of exercises marked in red could be used as a Chapter Test.

In Exercises 1 and 2, find the first four terms and the fortieth term of the given sequence.

1. $a_n = (-1)^n \frac{n+1}{n+3}$ for all $n \geq 1$ $-1/2, 3/5, -2/3, 5/7; a_{40} = 41/43$
2. $a_1 = -3, a_n = 2a_{n-1}$ for all $n \geq 2$ $-3, -6, -12, -24;$ $a_{40} = -3(2^{39})$
3. The sequence $-1, 1/2, 2, 7/2, \ldots$ is arithmetic. Find **(a)** the common difference, **(b)** the tenth term, and **(c)** an explicit rule for the nth term. **(a)** 3/2 **(b)** 25/2 **(c)** $a_n = \frac{3n-5}{2}$
4. The sequence $1/2, -2, 8, -32, \ldots$ is geometric. Find **(a)** the common ratio, **(b)** the seventh term, and **(c)** an explicit rule for the nth term. **(a)** -4 **(b)** 2048 **(c)** $a_n = (-1)^{n-1}(2^{2n-3})$

In Exercises 5 and 6, draw a graph of the sequence with given nth term.

5. $a_n = \frac{2^{n+1} + (-1)^n}{2^n}, n = 1, 2, 3, \ldots$
6. $a_n = (-1)^{n-1} \frac{n-1}{n}$

In Exercises 7 and 8, determine the convergence or divergence of the sequence with given nth term. If the sequence converges, find its limit.

7. $a_n = \frac{3n^2 - 1}{2n^2 + 1}$ converges, 3/2
8. $a_n = (-1)^n \frac{3n-1}{n+2}$ diverges

In Exercises 9–22, find the limit.

9. $\lim_{t \to 0} \frac{t - \ln(1 + 2t)}{t^2}$
10. $\lim_{t \to 0} \frac{\tan 3t}{\tan 5t}$ 3/5
11. $\lim_{x \to 0} \frac{x \sin x}{1 - \cos x}$ 2
12. $\lim_{x \to 1} x^{1/(1-x)}$ $1/e$
13. $\lim_{x \to \infty} x^{1/x}$ 1
14. $\lim_{x \to \infty} \left(1 + \frac{3}{x}\right)^x$ e^3
15. $\lim_{r \to \infty} \frac{\cos r}{\ln r}$ 0
16. $\lim_{\theta \to \pi/2} \left(\theta - \frac{\pi}{2}\right) \sec \theta$ -1
17. $\lim_{x \to 1} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right)$ $-1/2$
18. $\lim_{x \to 0^+} \left(1 + \frac{1}{x}\right)^x$ 1

9. The limit doesn't exist.

23. Same rate, because $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{1}{5}$ **24.** Same rate, because $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\ln 3}{\ln 2}$

19. $\lim_{\theta\to 0^+} (\tan\theta)^\theta$ 1

20. $\lim_{\theta\to\infty} \theta^2 \sin\left(\frac{1}{\theta}\right)$ ∞

21. $\lim_{x\to\infty} \frac{x^3 - 3x^2 + 1}{2x^2 + x - 3}$ ∞

22. $\lim_{x\to\infty} \frac{3x^2 - x + 1}{x^4 - x^3 + 2}$ 0

In Exercises 23–34, determine whether f grows faster than, slower than, or at the same rate as g as $x\to\infty$. Give reasons for your answer.

23. $f(x) = x, \quad g(x) = 5x$

24. $f(x) = \log_2 x, \quad g(x) = \log_3 x$

25. $f(x) = x, \quad g(x) = x + \frac{1}{x}$

26. $f(x) = \frac{x}{100}, \quad g(x) = xe^{-x}$

27. $f(x) = x, \quad g(x) = \tan^{-1} x$

28. $f(x) = \csc^{-1} x, \quad g(x) = \frac{1}{x}$

29. $f(x) = x^{\ln x}, \quad g(x) = x^{\log_2 x}$

30. $f(x) = 3^{-x}, \quad g(x) = 2^{-x}$

31. $f(x) = \ln 2x, \quad g(x) = \ln x^2$ Same rate, because $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{1}{2}$

32. $f(x) = 10x^3 + 2x^2, \quad g(x) = e^x$ Slower, because $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$

33. $f(x) = \tan^{-1} \frac{1}{x}, \quad g(x) = \frac{1}{x}$ Same rate, because $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$

34. $f(x) = \sin^{-1} \frac{1}{x}, \quad g(x) = \frac{1}{x^2}$ Faster, because $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$

In Exercises 35 and 36,

(a) show that f has a removable discontinuity at $x = 0$.

(b) define f at $x = 0$ so that it is continuous there.

35. $f(x) = \frac{2^{\sin x} - 1}{e^x - 1}$

36. $f(x) = x \ln x$

In Exercises 37–48, evaluate the improper integral or state that it diverges.

37. $\int_1^\infty \frac{dx}{x^{3/2}}$ 2

38. $\int_1^\infty \frac{dx}{x^2 + 7x + 12}$ ln (5/4)

39. $\int_{-\infty}^{-1} \frac{3dx}{3x - x^2}$ $-2\ln(2)$

40. $\int_0^3 \frac{dx}{\sqrt{9 - x^2}}$ $\pi/2$

41. $\int_0^1 \ln(x)\,dx$ -1

42. $\int_{-1}^1 \frac{dy}{y^{2/3}}$ 6

43. $\int_{-2}^0 \frac{d\theta}{(\theta + 1)^{3/5}}$ 0

44. $\int_3^\infty \frac{2dx}{x^2 - 2x}$ ln (3)

45. $\int_0^\infty x^2 e^{-x}\,dx$ 2

46. $\int_{-\infty}^0 xe^{3x}\,dx$ $-1/9$

47. $\int_{-\infty}^\infty \frac{dx}{e^x + e^{-x}}$ $\pi/2$

48. $\int_{-\infty}^\infty \frac{4dx}{x^2 + 16}$ π

In Exercises 49 and 50, use the comparison test to determine whether the improper integral converges or diverges.

49. $\int_1^\infty \frac{\ln z}{z}\,dz$ diverges

50. $\int_1^\infty \frac{e^{-t}}{\sqrt{t}}\,dt$ converges

51. The second and fifth terms of a geometric sequence are -3 and $-3/8$, respectively. Find **(a)** the first term, **(b)** the common ratio, and **(c)** an explicit formula for the nth term. **(a)** -6 **(b)** 1/2 **(c)** $a_n = -3(2^{2-n})$

52. **(a)** 13 **(b)** -1.5 **(c)** $a_n = -1.5n + 14.5$

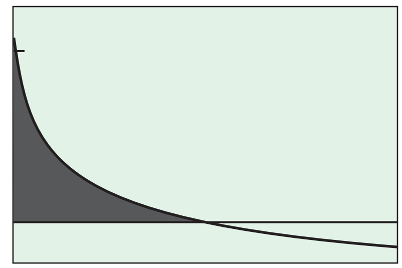
52. The second and sixth terms of an arithmetic sequence are 11.5 and 5.5, respectively. Find **(a)** the first term, **(b)** the common difference, and **(c)** an explicit formula for the nth term.

53. Consider the improper $\int_{-\infty}^{\infty} e^{-2|x|}\,dx$.

(a) Express the improper integral as a limit of definite integrals.

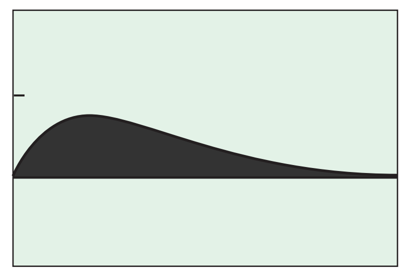
(b) Evaluate the integral. **(a)** $\lim_{b\to-\infty} \int_b^0 e^{2x}\,dx + \lim_{b\to\infty} \int_0^b e^{-2x}\,dx$ **(b)** 1

54. ***Infinite Solid*** The infinite region bounded by the coordinate axes and the curve $y = -\ln x$ in the first quadrant (see figure) is revolved about the x-axis to generate a solid. Find the volume of the solid. 2π



[0, 2] by [–1, 5]

55. ***Infinite Region*** Find the area of the region in the first quadrant under the curve $y = xe^{-x}$ (see figure). 1



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AP* Examination Preparation

You should solve the following problems without using a graphing calculator.

56. Consider the infinite region R in the first quadrant under the curve $y = xe^{-x/2}$.

(a) Write the area of R as an improper integral.

(b) Express the integral in part (a) as a limit of a definite integral.

(c) Find the area of R.

57. The infinite region in the first quadrant bounded by the coordinate axes and the curve $y = \frac{1}{x} - 1$ is revolved about the y-axis to generate a solid.

(a) Write the volume of the solid as an improper integral.

(b) Express the integral in part (a) as a limit of a definite integral.

(c) Find the volume of the solid.

58. Determine whether or not $\int_0^\infty xe^{-x}\,dx$ converges. If it converges, give its value. Show your reasoning.

25. Same rate, because $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ **27.** Faster, because $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$

26. Faster, because $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$ **28.** Same rate, because $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$

29. Slower, because $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$ **30.** Slower, because $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$