

Let p and q be real numbers and let f be the function defined by:

$$f(x) = \begin{cases} 1 + 2p(x-1) + (x-1)^2, & \text{for } x \leq 1 \\ qx + p, & \text{for } x > 1. \end{cases}$$

$$\begin{aligned} 1 + 2px - 2p + x^2 - 2x + 1 \\ 1 + \frac{2}{3}x - \frac{4}{3} + x^2 - 2x + 1 \end{aligned}$$

- (a) Find the value of q , in terms of p , for which f is continuous at $x=1$.

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = f(1)$$

$$1 + 2p(1-1) + (1-1)^2 = q(1) + p$$

$$1 = q + p$$

$$\boxed{q = 1 - p}$$

- (b) Find the values of p and q for which f is differentiable at $x=1$.

$$\begin{aligned} \lim_{x \rightarrow 1^-} f'(x) &= \lim_{x \rightarrow 1^+} f'(x) \\ \lim_{x \rightarrow 1^-} 2p + 2x - 2 &= q \\ 2p = q & \quad \begin{aligned} 2p &= 1 - p \\ 3p &= 1 \\ p &= \frac{1}{3} \end{aligned} \quad \begin{aligned} q &= 2p \\ q &= \frac{2}{3} \end{aligned} \end{aligned}$$

- (c) If p and q have the values determined in part (b), is f'' a continuous function? Justify your answer.

$$f(x) = 1 + \frac{2}{3}x - \frac{4}{3} + x^2 - 2x + 1 \quad \frac{2}{3}x + \frac{1}{3}$$

$$f'(x) = \frac{2}{3} + 2x - 2 \quad \frac{2}{3}$$

$$f''(x) = 2 \neq 0$$

Since $\lim_{x \rightarrow 1^-} f''(x) \neq \lim_{x \rightarrow 1^+} f''(x)$, $f''(x)$ is not cont. num at $x=1$

Let f be the function defined by $f(x) = x^4 - 3x^2 + 2$. Let $u = x^2$, then $f(u) = u^2 - 3u + 2 = 0$

- (a) Find the zeros of f .

$$\begin{aligned} u = x^2 = 1 &\quad u = x^2 = 2 \\ x = \pm 1 &\quad x = \pm \sqrt{2} \end{aligned} \quad \begin{aligned} u = 2 &\quad u = 1 \\ u = 2 &\quad u = 1 \end{aligned}$$

- (b) Write an equation of the line tangent to the graph of f at the point where $x=1$.

$$(x, f(x)) = (1, f(1)) = (1, 0) \quad f'(x) = 4x^3 - 6x \quad f'(1) = 4 \cdot 1 - 6 = -2$$

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -2(x - 1) \Rightarrow y = -2(x - 1) = -2x + 2$$

- (c) Find the x -coordinate of each point at which the line tangent to the graph of f is parallel to the line $y = -2x + 4$. Since $m = -2$, find all x such that $f'(x) = -2$

$$\begin{aligned} f'(x) &= 4x^3 - 6x = -2 \quad \begin{array}{|c c c c|} \hline & 1 & 2 & 0 & -3 & 1 \\ & & 2 & 2 & 2 & -1 \\ \hline & 2 & 2 & 2 & 0 & 0 \\ \hline \end{array} \quad D = b^2 - 4ac \\ 4x^3 - 6x + 2 &= 0 \\ 2(2x^3 - 3x + 1) &= 0 \\ 2x^3 - 3x + 1 &= 0 \end{aligned}$$

$$\boxed{x=1}$$

$$\begin{aligned} &= 4 - 4(2)(-1) \\ &= 12 \\ \sqrt{D} &= \sqrt{12} = 2\sqrt{3} \quad \begin{aligned} x &= \frac{-b \pm \sqrt{D}}{2a} \\ x &= \frac{1 \pm \sqrt{3}}{2} \end{aligned} \end{aligned}$$

Let f be the real-valued function defined by $f(x) = \sin^3 x + \sin^3|x|$.

$$\begin{aligned}
 \text{(a) Find } f'(x) \text{ for } x > 0. & \quad \text{remember: } \sin(-x) = -\sin(x) \\
 & \quad \begin{cases} \sin^3 x + \sin^3 x, & x > 0 \\ \sin^3 x + \sin^3(-x), & x < 0 \end{cases} \\
 \text{(b) Find } f'(x) \text{ for } x < 0. & \quad \begin{cases} 2\sin^3 x, & x > 0 \\ 0, & x < 0 \end{cases} \\
 f'(x) = \begin{cases} \begin{array}{l} \text{a)} \\ 6\sin^2 x \cos x, \\ 0 \end{array}, & x > 0 \\ \begin{array}{l} \text{b)} \\ 0 \end{array}, & x < 0 \end{cases}
 \end{aligned}$$

(c) Determine whether $f(x)$ is continuous at $x = 0$. Justify your answer.

$$\begin{aligned}
 \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^+} f(x) = f(0) \\
 0 &\stackrel{?}{=} 0 \quad \therefore \text{yes, continuous}
 \end{aligned}$$

(d) Determine whether the derivative of $f(x)$ exists at $x = 0$. Justify your answer.

$$\begin{aligned}
 \text{derivative exists if,} \\
 \lim_{x \rightarrow 0^-} f'(x) = 0 \quad \text{is equal to} \quad \lim_{x \rightarrow 0^+} f'(x) = 0 \quad \therefore \text{derivative exists!}
 \end{aligned}$$

$$\begin{aligned}
 \text{Let } f \text{ be the real-valued function defined by } f(x) = \sqrt{1+6x}. & \quad (1+6x)^{1/2} \\
 \text{Domain: } & \quad \begin{aligned} 1+6x &\geq 0 \\ 6x &\geq -1 \\ x &\geq -\frac{1}{6} \end{aligned} \quad \text{Range: } [0, \infty)
 \end{aligned}$$

(a) Give the domain and range of f .

$$\begin{aligned}
 \text{(b) Determine the slope of the line tangent to the graph of } f \text{ at } x = 4. & \quad \text{find } f'(4) \\
 f'(x) = \frac{1}{2}(1+6x)^{-1/2}(6) = 3(1+6x)^{-1/2} = \frac{3}{\sqrt{1+6x}}, & \quad \begin{cases} f'(4) = \frac{3}{\sqrt{25}} = \frac{3}{5} \end{cases}
 \end{aligned}$$

(c) Determine the y -intercept of the line tangent to the graph of f at $x = 4$.

$$\begin{aligned}
 x=4 \quad y=5 \quad m=\frac{3}{5} \quad y-5 = \frac{3}{5}(x-4) \Rightarrow y = \frac{3}{5}x - \frac{12}{5} + \frac{25}{5} = \frac{3}{5}x + \frac{13}{5} \\
 \therefore y\text{-intercept is } \frac{13}{5}
 \end{aligned}$$

(d) Give the coordinates of the point on the graph of f where the tangent line is parallel to $y = x + 12$. Since $m=1$, find $f'(x)=1$

$$\begin{aligned}
 f'(x) = \frac{3}{\sqrt{1+6x}} = \frac{1}{1} & \quad \begin{aligned} 1+6x &= 9 \\ 6x &= 8 \\ x &= 8/6 = 4/3 \end{aligned} \\
 (\sqrt{1+6x})^2 = (3)^2 & \quad \begin{aligned} \frac{3}{\sqrt{1+6(\frac{4}{3})}} &= \frac{3}{\sqrt{1+8}} = \frac{3}{\sqrt{9}} = \frac{3}{3} = 1 \end{aligned} \\
 & \quad \therefore f'(x)=1 \text{ when } x=\frac{4}{3}
 \end{aligned}$$

Let f be the function defined by $f(x) = \frac{2x-5}{x^2-4}$.

all reals where

(a) Find the domain of f . $x \neq 2$ $x \neq -2$

(b) Write an equation for each vertical and each horizontal asymptote for the graph of f .

$$\text{Vertical: } x=2 \quad x=-2 \quad \text{Horizontal: } \lim_{x \rightarrow \infty} \frac{2x-5}{x^2-4} = 0 \quad \lim_{x \rightarrow -\infty} \frac{2x-5}{x^2-4} = 0$$

so $y=0$ is the horizontal asymptote

(c) Find $f'(x)$.

$$f'(x) = \frac{(x^2-4)(2) - (2x-5)(2x)}{(x^2-4)^2} = \frac{2x^2-8-4x^2+10x}{(x^2-4)^2} = \frac{-2x^2+10x-8}{(x^2-4)^2}$$

$$= \frac{-2(x^2-5x+4)}{(x^2-4)^2}$$

$$= \frac{-2(x-4)(x-1)}{(x^2-4)^2}$$

$x=0 \quad y=\frac{5}{4} \quad m=-\frac{1}{2}$

$y-\frac{5}{4} = -\frac{1}{2}x$ or $y = -\frac{1}{2}x + \frac{5}{4}$

(d) Write an equation for the line tangent to the graph of f at the point $(0, f(0))$.

Let $f(x) = \sqrt{1-\sin x}$.

defined when

$$\begin{aligned} 1-\sin x &\geq 0 \\ \sin x &\leq 1 \end{aligned}$$

always true

(a) What is the domain of f ? all reals

(b) Find $f'(x)$.

$$f'(x) = \frac{1}{2}(1-\sin x)^{-1/2}(-\cos x) = \frac{-\cos x}{2\sqrt{1-\sin x}}$$

(c) What is the domain of f' ? Not defined when

$$\begin{aligned} 1-\sin x &= 0 \\ \sin x &= 1 \\ \text{i.e. } x &= \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \dots \end{aligned}$$

(d) Write an equation for the line tangent to the graph of f at $x=0$.

$$x=0 \quad y=1 \quad m=-\frac{1}{2}$$

$$y-1 = -\frac{1}{2}x$$

$$y = -\frac{1}{2}x + 1$$

Let f be the function given by $f(x) = \sqrt{x^4 - 16x^2}$.

(a) Find the domain of f . $(-\infty, -2] \cup [2, \infty)$

(b) Describe the symmetry, if any, of the graph of f .

Even if $f(-x) = f(x)$
(symmetric about y -axis)

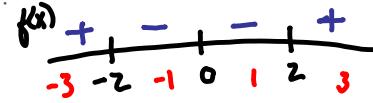
odd if $f(-x) = -f(x)$
(symmetric about origin)
 \therefore function is even and symmetric about the y -axis

(c) Find $f'(x)$.

$$f'(x) = \frac{1}{2}(x^4 - 16x^2)^{-\frac{1}{2}} (4x^3 - 32x)$$

$$= \frac{4x(x^2 - 8)}{2\sqrt{x^4 - 16x^2}} = \boxed{\frac{2x(x^2 - 8)}{\sqrt{x^4 - 16x^2}}}$$

$$\begin{aligned} x^4 - 16x^2 &> 0 \\ x^2(x^2 - 16) &\geq 0 \\ x^2(x+4)(x-4) &\geq 0 \end{aligned}$$



$$f'(5) = \frac{10(17)}{\sqrt{625 - 400}} = \frac{170}{15} = \frac{34}{3}$$

$$m_{\perp} = -\frac{3}{34}$$

(d) Find the slope of the line normal to the graph of f at $x = 5$.

A particle moves along the x -axis in such a way that its position at time t for $t \geq 0$ is

$$\text{given by } x = \frac{1}{3}t^3 - 3t^2 + 8t. \quad x(t) = \frac{1}{3}(t^3) - 3(t^2) + 8(t) = \frac{8}{3} - 12 + 16 = \frac{8}{3} + 4 = \frac{8}{3} + \frac{12}{3} = \frac{20}{3}$$

(a) Show that at time $t = 0$, the particle is moving to the right.

$$v(t) = t^2 - 6t + 8 \text{ and } v(0) = 8$$

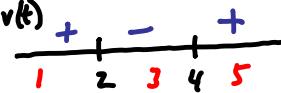
moving right when $v(t) > 0$,
so show $v(0) > 0$.

Since $v(0) = 8$, moving right at $t=0$

(b) Find all values of t for which the particle is moving to the left.

$$v(t) = t^2 - 6t + 8 = (t-4)(t-2) = 0 \text{ when } t=4, t=2$$

For $2 < t < 4$, particle moves left.



(c) What is the position of the particle at time $t = 3$?

$$x(3) = \frac{1}{3}(3)^3 - 3(3)^2 + 8(3) = -18 + 24 = 6$$

(d) When $t = 3$, what is the total distance the particle has traveled?

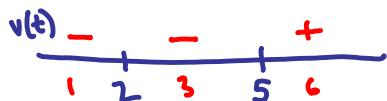
$$6\frac{2}{3} + 2\frac{2}{3} = 7\frac{1}{3} = 22\frac{1}{3}$$

t	$x(t)$
0	0
2	$20/3 = 6\frac{2}{3}$
3	6

A particle starts at time $t = 0$ and moves on a number line so that its position at time t is given by $x(t) = (t-2)^3(t-6)$.

- (a) When is the particle moving to the right? when $v(t) > 0$, i.e. $t > 5$
- (b) When is the particle at rest? when $v(t) = 0$, i.e. when $t=2$ or $t=5$
- (c) When does the particle change direction? when $v(t)$ changes sign, i.e. at $t=5$

$$\begin{aligned} v(t) &= (t-2)^3(1) + (t-6)3(t-2)^2(1) \\ &= (t-2)^3 + 3(t-6)(t-2)^2 \\ &= (t-2)^2[t-2+3(t-6)] \\ &= (t-2)^2(4t-20) \\ &= 4(t-2)^2(t-5) \end{aligned}$$



- (d) What is the farthest to the left of the origin that the particle moves?
 Since $x(0) = 48$ and $x(5) = -27$
 and $v(t) < 0$ for $0 < t < 5$ and $v(t) > 0$ for $t > 5$ then $x = -27$ is the farthest left of the origin that the particle moves

A particle moves along a line so that at any time t its position is given by
 $x(t) = 2\pi t + \cos 2\pi t$.

- (a) Find the velocity at time t . $v(t) = 2\pi - \sin(2\pi t) \cdot 2\pi = 2\pi(1 - \sin 2\pi t)$

- (b) Find the acceleration at time t . $a(t) = 4\pi^2 \cos(2\pi t)$

- (c) What are all values of t , $0 \leq t \leq 3$, for which the particle is at rest?

$$\begin{aligned} \text{when } v(t) = 0, \text{ or whenever } 1 - \sin 2\pi t = 0 \text{ or } \sin 2\pi t = 1 \text{ when } t = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \dots \\ 2\pi t = \frac{\pi}{2} \quad 2\pi t = \frac{5\pi}{2} \quad 2\pi t = \frac{9\pi}{2} \quad 2\pi t = \frac{13\pi}{2} \\ t = \frac{1}{4} \quad t = \frac{5}{4} \quad t = \frac{9}{4} \quad t = \frac{13}{4} \end{aligned}$$

- (d) What is the maximum velocity?
 find $a(t) = 0$ at $t = 0$ where $\cos(2\pi t) = 0$ $\cos x = 0$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$

$$\begin{array}{ccccccccc} 2\pi t = \frac{\pi}{2} & 2\pi t = \frac{3\pi}{2} & 2\pi t = \frac{5\pi}{2} & & & & & & \\ t = \frac{1}{4} & t = \frac{3}{4} & t = \frac{5}{4} & & & & & & \\ t = \frac{9}{4} & t = \frac{11}{4} & & & & & & & \\ \hline v(t) & 0 & \frac{1}{4}\pi & 0 & \frac{5}{4}\pi & 0 & \frac{9}{4}\pi & 0 & \frac{13}{4}\pi \end{array}$$

Max velocity is $\frac{13}{4}\pi$.