

**AP<sup>®</sup> CALCULUS BC  
2004 SCORING GUIDELINES**

Question 6

Let  $f$  be the function given by  $f(x) = \sin\left(5x + \frac{\pi}{4}\right)$ , and let  $P(x)$  be the third-degree Taylor polynomial for  $f$  about  $x = 0$ .

(a) Find  $P(x)$ .

$$f(x) = \sin\left(5x + \frac{\pi}{4}\right)$$

$$f'(x) = \cos\left(5x + \frac{\pi}{4}\right)$$

$$f''(x) = (5)^2 \sin\left(5x + \frac{\pi}{4}\right)$$

$$f'''(x) = -(5)^3 \cos\left(5x + \frac{\pi}{4}\right)$$

$$f^{(4)}(x) = (5)^4 \sin\left(5x + \frac{\pi}{4}\right)$$

$$f(0) = \frac{5\sqrt{2}}{2}$$

$$f'(0) = \frac{5'\sqrt{2}}{2}$$

$$f''(0) = -\frac{5^2\sqrt{2}}{2}$$

$$f'''(0) = -\frac{5^3\sqrt{2}}{2}$$

$$f^{(4)}(0) = \frac{5^4\sqrt{2}}{2}$$

$$P(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!}$$

$$= \frac{5^0\sqrt{2}}{2} + \frac{5^1\sqrt{2}}{2}x - \frac{5^2\sqrt{2}}{2 \cdot 2!}x^2 - \frac{5^3\sqrt{2}}{2 \cdot 3!}x^3$$

0      1      2      3

$$a_n = \boxed{\frac{5^n\sqrt{2}}{2 \cdot n!} x^n}$$

$$\boxed{-\frac{5^{22}\sqrt{2}}{2 \cdot 22!} x^{22}}$$

(b) Find the coefficient of  $x^{22}$  in the Taylor series for  $f$  about  $x = 0$ .

01 23 45 67 89 101 1213 1415 1617 1819 2021 2223

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(c) Use the Lagrange error bound to show that  $|f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right)| < \frac{1}{100}$ .

$$\left| f(x) - P_n(x) \right| < \left| \frac{f^{n+1}(c) x^{n+1}}{(n+1)!} \right| \quad \begin{array}{l} \text{where } c \text{ is between } a \text{ and } x \\ \text{here } x = \frac{1}{10} \quad a = 0 \end{array}$$

$$< \left| \frac{5^{n+1} x^{n+1}}{(n+1)!} \right|$$

$$< \left| \frac{5 x^4}{4!} \right|$$

$$< \frac{625}{10^4 \cdot 4!} < \frac{1000}{24000} < \frac{1}{240} < \frac{1}{100}$$

$$f^{(4)}(x) = (5)^4 \sin\left(5x + \frac{\pi}{4}\right)$$

$$\sin\left(5x + \frac{\pi}{4}\right) = 1 \text{ when } 5x + \frac{\pi}{4} = \frac{\pi}{2}$$

$$5x = \frac{\pi}{4}$$

$$x = \frac{\pi}{20} \approx \frac{1}{7}$$

(d) Let  $G$  be the function given by  $G(x) = \int_0^x f(t) dt$ . Write the third-degree Taylor polynomial for  $G$  about  $x = 0$ .

$$f(x) \approx \frac{5^0\sqrt{2}}{2} + \frac{5^1\sqrt{2}}{2}x - \frac{5^2\sqrt{2}}{2 \cdot 2!}x^2 - \frac{5^3\sqrt{2}}{2 \cdot 3!}x^3$$

$$G(x) = \int_0^x f(t) dt \approx \frac{\sqrt{2}}{2}x + \frac{5\sqrt{2}}{2 \cdot 2}x^2 - \frac{5^2\sqrt{2}}{2 \cdot 3 \cdot 2!}x^3$$

$$= \frac{\sqrt{2}}{2}x + \frac{5\sqrt{2}}{4}x^2 - \frac{25\sqrt{2}}{12}x^3$$

(a)  $f(0) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$   
 $f'(0) = 5\cos\left(\frac{\pi}{4}\right) = \frac{5\sqrt{2}}{2}$   
 $f''(0) = -25\sin\left(\frac{\pi}{4}\right) = -\frac{25\sqrt{2}}{2}$   
 $f'''(0) = -125\cos\left(\frac{\pi}{4}\right) = -\frac{125\sqrt{2}}{2}$   
 $P(x) = \frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}x - \frac{25\sqrt{2}}{2(2!)}x^2 - \frac{125\sqrt{2}}{2(3!)}x^3$

(b)  $\frac{-5^{21}\sqrt{2}}{2(22!)}$

(c) 
$$\begin{aligned} \left| f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right) \right| &\leq \max_{0 \leq c \leq \frac{1}{10}} |f^{(4)}(c)| \left| \left(\frac{1}{4!}\right) \left(\frac{1}{10}\right)^4 \right| \\ &\leq \frac{625}{4!} \left(\frac{1}{10}\right)^4 = \frac{1}{384} < \frac{1}{100} \end{aligned}$$

(d) The third-degree Taylor polynomial for  $G$  about  $x = 0$  is 
$$\begin{aligned} &\int_0^x \left( \frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}t - \frac{25\sqrt{2}}{4}t^2 \right) dt \\ &= \frac{\sqrt{2}}{2}x + \frac{5\sqrt{2}}{4}x^2 - \frac{25\sqrt{2}}{12}x^3 \end{aligned}$$

4 :  $P(x)$   
 $\langle -1 \rangle$  each error or missing term  
deduct only once for  $\sin\left(\frac{\pi}{4}\right)$   
evaluation error  
deduct only once for  $\cos\left(\frac{\pi}{4}\right)$   
evaluation error  
 $\langle -1 \rangle$  max for all extra terms,  $+ \dots$ ,  
misuse of equality

2 :  $\begin{cases} 1 : \text{magnitude} \\ 1 : \text{sign} \end{cases}$

1 : error bound in an appropriate inequality

2 : third-degree Taylor polynomial for  $G$  about  $x = 0$   
 $\langle -1 \rangle$  each incorrect or missing term  
 $\langle -1 \rangle$  max for all extra terms,  $+ \dots$ ,  
misuse of equality