$f''(2) = \frac{1!}{3^2}$ $f^{(4)}(2) = \frac{3!}{3^4}$

$f^{(6)}(2) = \frac{5!}{3^6}$

Question 6

Let f be a function with derivatives of all orders and for which $f(2) = 7$. When n is odd, the nth derivative of f at $x = 2$ is 0. When n is even and $n \geq 2$, the nth derivative of f at $x = 2$ is given by $f^{(n)}(2) = \frac{(n-1)!}{3^n}$.

(a) Write the sixth-degree Taylor polynomial for f about $x = 2$.

$$T_6(x) = \underset{0}{f(2)} + \underset{1}{\cancel{f'(2)(x-2)}} + \underset{2}{\frac{f''(2)(x-2)^2}{2!}} + \underset{3}{\cancel{\frac{f'''(2)(x-2)^3}{3!}}} + \underset{4}{\frac{f^{(4)}(2)(x-2)^4}{4!}} + \underset{5}{\cancel{\frac{f^{(5)}(2)(x-2)^5}{5!}}} + \underset{6}{\frac{f^{(6)}(2)(x-2)^6}{6!}}$$

$$= f(2) + \frac{f''(2)(x-2)^2}{2!} + \frac{f^{(4)}(2)(x-2)^4}{4!} + \frac{f^{(6)}(2)(x-2)^6}{6!}$$

$$= 7 + \frac{1!\,(x-2)^2}{3^2 \cdot 2!} + \frac{3!\,(x-2)^4}{3^4 \cdot 4!} + \frac{5!\,(x-2)^6}{3^6 \cdot 6!}$$

$$= \underset{0}{7} + \underset{1}{\frac{(x-2)^2}{3^2 \cdot 2}} + \underset{2}{\frac{(x-2)^4}{3^4 \cdot 4}} + \underset{3}{\frac{(x-2)^6}{3^6 \cdot 6}}$$

$$\boxed{\frac{(x-2)^{2n}}{3^{2n} \cdot 2n}}$$

(b) In the Taylor series for f about $x = 2$, what is the coefficient of $(x-2)^{2n}$ for $n \geq 1$?

$$\boxed{\frac{1}{3^{2n} \cdot 2n}}$$

(c) Find the interval of convergence of the Taylor series for f about $x = 2$. Show the work that leads to your answer.

Two ways to find an interval of convergence:
For geometric series, $|r| < 1$
Otherwise $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$
With the ratio test, don't forget to check the endpoints

$$a_n = \frac{(x-2)^{2n}}{3^{2n} \cdot 2n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{2n+2}}{3^{2n+2}(2n+2)}\right| \cdot \left|\frac{3^{2n} \cdot 2n}{(x-2)^{2n}}\right| = \lim_{n\to\infty} \frac{1}{3^2} \cdot (x-2)^2 = \frac{(x-2)^2}{9} < 1$$

$$(x-2)^2 < 9 \rightarrow \sqrt{(x-2)^2} < \sqrt{9} \rightarrow |x-2| < 3 \rightarrow -3 < x-2 < 3$$

$$\boxed{-1 < x < 5}$$

@ $x=-1$ $\quad \frac{3^{2n}}{3^{2n} \cdot 2n} = \frac{1}{2n}$ diverges because $\frac{1}{n}$ diverges

@ $x=5$ $\quad \frac{3^{2n}}{3^{2n} \cdot 2n} = \frac{1}{2n}$ diverges by same LCT

$\lim_{n\to\infty} \frac{1}{2n} \cdot \frac{n}{1} = \frac{1}{2}$

LCT says when $\lim_{n\to\infty} \frac{a_n}{b_n} \in \mathbb{R}$ both converge or both diverge

(a) $P_6(x) = 7 + \frac{1!}{3^2}\cdot\frac{1}{2!}(x-2)^2 + \frac{3!}{3^4}\cdot\frac{1}{4!}(x-2)^4 + \frac{5!}{3^6}\cdot\frac{1}{6!}(x-2)^6$

$$3 : \begin{cases} 1 : \text{polynomial about } x = 2 \\ 2 : P_6(x) \\ \quad \langle -1 \rangle \text{ each incorrect term} \\ \quad \langle -1 \rangle \text{ max for all extra terms,} \\ \qquad + \cdots, \text{ misuse of equality} \end{cases}$$

(b) $\frac{(2n-1)!}{3^{2n}}\cdot\frac{1}{(2n)!} = \frac{1}{3^{2n}(2n)}$

1 : coefficient

(c) The Taylor series for f about $x = 2$ is

$$f(x) = 7 + \sum_{n=1}^{\infty}\frac{1}{2n\cdot 3^{2n}}(x-2)^{2n}.$$

$$L = \lim_{n\to\infty}\left|\frac{\frac{1}{2(n+1)}\cdot\frac{1}{3^{2(n+1)}}(x-2)^{2(n+1)}}{\frac{1}{2n}\cdot\frac{1}{3^{2n}}(x-2)^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2n}{2(n+1)}\cdot\frac{3^{2n}}{3^2 3^{2n}}(x-2)^2\right| = \frac{(x-2)^2}{9}$$

$L < 1$ when $|x-2| < 3$.

Thus, the series converges when $-1 < x < 5$.

When $x = 5$, the series is $7 + \sum_{n=1}^{\infty}\frac{3^{2n}}{2n\cdot 3^{2n}} = 7 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$,

which diverges, because $\sum_{n=1}^{\infty}\frac{1}{n}$, the harmonic series, diverges.

When $x = -1$, the series is $7 + \sum_{n=1}^{\infty}\frac{(-3)^{2n}}{2n\cdot 3^{2n}} = 7 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$,

which diverges, because $\sum_{n=1}^{\infty}\frac{1}{n}$, the harmonic series, diverges.

The interval of convergence is $(-1, 5)$.

$$5 : \begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{identifies interior of} \\ \quad \text{interval of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis/conclusion for} \\ \quad \text{both endpoints} \end{cases}$$