

Question 6

The function f is defined by $f(x) = \frac{1}{1+x^3}$. The Maclaurin series for f is given by

$$1 - x^3 + x^6 - x^9 + \dots + (-1)^n x^{3n} + \dots,$$

which converges to $f(x)$ for $-1 < x < 1$.

- (a) Find the first three nonzero terms and the general term for the Maclaurin series for $f'(x)$.

$$f'(x) = -3x^2 + 6x^5 - 9x^8 + \dots + (-1)^n 3n x^{3n-1} \quad \text{starts with } n=1$$

- (b) Use your results from part (a) to find the sum of the infinite series $\frac{3}{2^2} + \frac{6}{2^5} - \frac{9}{2^8} + \dots + (-1)^n \frac{3n}{2^{3n-1}} + \dots$. $x = \frac{1}{2}$

$$\begin{aligned} f(x) &= \frac{1}{1+x^3} & f'(x) &= -(1+x^3)^{-2}(3x^2) & f'\left(\frac{1}{2}\right) &= -\frac{\frac{3}{4}}{\left(1+\frac{1}{8}\right)^2} \\ &= (1+x^3)^{-1} & &= \frac{-3x^2}{(1+x^3)^2} & &= -\frac{3}{4} \cdot \frac{64}{81} = \boxed{-\frac{16}{27}} \end{aligned}$$

- (c) Find the first four nonzero terms and the general term for the Maclaurin series representing $\int_0^x f(t) dt$.

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which converges to $f(x)$ for $-1 < x < 1$.

$$\int_0^x f(t) dt = x - \frac{x^4}{4} + \frac{x^7}{7} + \dots + \frac{(-1)^n x^{3n+1}}{3n+1} + \dots \quad \text{start with } n=0$$

- (d) Use the first three nonzero terms of the infinite series found in part (c) to approximate $\int_0^{1/2} f(t) dt$. What are the properties of the terms of the series representing $\int_0^{1/2} f(t) dt$ that guarantee that this approximation is

Alternating series converges when:

1. strictly alternates

2. $|a_{n+1}| < |a_n|$

3. $\lim_{n \rightarrow \infty} a_n = 0$

$$\int_0^{1/2} f(t) dt \approx \left(\frac{1}{2}\right) - \frac{\left(\frac{1}{2}\right)^4}{4} + \frac{\left(\frac{1}{2}\right)^7}{7} = \frac{1}{2} - \frac{1}{64} + \frac{1}{896} = \frac{448}{896} - \frac{14}{896} + \frac{1}{896} = \boxed{\frac{435}{896}}$$

when these criteria are met, the series converges and the magnitude of the error is less than the magnitude of the last unused term. $\frac{1}{10} = \frac{1}{10240} < \frac{1}{10,000}$

(a) $f'(x) = -3x^2 + 6x^5 - 9x^8 + \dots + 3n(-1)^n x^{3n-1} + \dots$

2 : $\begin{cases} 1 : \text{first three terms} \\ 1 : \text{general term} \end{cases}$

- (b) The given series is the Maclaurin series for $f'(x)$ with $x = \frac{1}{2}$.

2 : $\begin{cases} 1 : f'(x) \\ 1 : f'\left(\frac{1}{2}\right) \end{cases}$

$$f'(x) = -(1+x^3)^{-2}(3x^2)$$

$$\text{Thus, the sum of the series is } f'\left(\frac{1}{2}\right) = -\frac{3\left(\frac{1}{2}\right)}{\left(1+\frac{1}{8}\right)^2} = -\frac{16}{27}.$$

(c) $\int_0^x \frac{1}{1+t^3} dt = x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10} + \dots + \frac{(-1)^n x^{3n+1}}{3n+1} + \dots$

2 : $\begin{cases} 1 : \text{first four terms} \\ 1 : \text{general term} \end{cases}$

$$(d) \int_0^{1/2} \frac{1}{1+t^3} dt \approx \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^4}{4} + \frac{\left(\frac{1}{2}\right)^7}{7}.$$

3 : $\begin{cases} 1 : \text{approximation} \\ 1 : \text{properties of terms} \\ 1 : \text{absolute value of fourth term} \\ < 0.0001 \end{cases}$

The series in part (c) with $x = \frac{1}{2}$ has terms that alternate, decrease in absolute value, and have limit 0. Hence the error is bounded by the absolute value of the next term.

$$\left| \int_0^{1/2} \frac{1}{1+t^3} dt - \left(\frac{1}{2} - \frac{\left(\frac{1}{2}\right)^4}{4} + \frac{\left(\frac{1}{2}\right)^7}{7} \right) \right| < \frac{\left(\frac{1}{2}\right)^{10}}{10} = \frac{1}{10240} < 0.0001$$