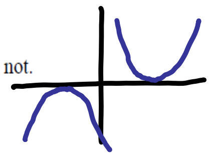
**Question 5**   $\frac{d}{dx}\left[-6(y-2)^{-1}\right] = 6(y-2)^{-2}\frac{dy}{dx}$

Consider the differential equation $\frac{dy}{dx} = 5x^2 - \frac{6}{y-2}$ for $y \neq 2$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(-1) = -4$.

(a) Evaluate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $(-1, -4)$.

$\left.\frac{dy}{dx}\right|_{(-1,-4)} = 5 - \frac{6}{-4-2} = 6$   $\left.\frac{d^2y}{dx^2}\right|_{(-1,-4)} = 10(-1) + \frac{6}{(-4-2)^2}(6) = -10+1 = -9$

(b) Is it possible for the $x$-axis to be tangent to the graph of $f$ at some point? Explain why or why not.



let $y=0$

$\frac{dy}{dx} = 5x^2 - \frac{6}{y-2} = 5x^2 + 3 \neq 0$ for any value of $x$

so x-axis cannot be tangent to f anywhere.

(c) Find the second-degree Taylor polynomial for $f$ about $x = -1$.   $T_2(x) = f(-1) + f'(-1)(x+1) + \frac{f''(-1)(x+1)^2}{2!}$

$T_2(x) = -4 + 6(x+1) - \frac{9}{2}(x+1)^2$   $\frac{dy}{dx} = 5x^2 - \frac{6}{y-2}$

(d) Use Euler's method, starting at $x = -1$ with two steps of equal size, to approximate $f(0)$. Show the work that leads to your answer.

| $x$ | $y$ | $\frac{dy}{dx}$ | $\Delta x$ | $\Delta y = \frac{dy}{dx} \cdot \Delta x$ | $x + \Delta x$ | $y + \Delta y$ |
|---|---|---|---|---|---|---|
| $-1$ | $-4$ | $6$ | $.5$ | $3$ | $-.5$ | $-1$ |
| $-.5$ | $-1$ | $\frac{5}{4}+2=\frac{13}{4}$ | $.5$ | $\frac{13}{8}$ | $0$ | $\frac{5}{8}$ |

$f(0) \approx \frac{5}{8}$

(a) $\left.\frac{dy}{dx}\right|_{(-1,-4)} = 6$

$\frac{d^2y}{dx^2} = 10x + 6(y-2)^{-2}\frac{dy}{dx}$

$\left.\frac{d^2y}{dx^2}\right|_{(-1,-4)} = -10 + 6\frac{1}{(-6)^2}6 = -9$

$3: \begin{cases} 1: \left.\frac{dy}{dx}\right|_{(-1,-4)} \\ 1: \frac{d^2y}{dx^2} \\ 1: \left.\frac{d^2y}{dx^2}\right|_{(-1,-4)} \end{cases}$

(b) The $x$-axis will be tangent to the graph of $f$ if $\left.\frac{dy}{dx}\right|_{(k,0)} = 0.$

The $x$-axis will never be tangent to the graph of $f$ because $\left.\frac{dy}{dx}\right|_{(k,0)} = 5k^2 + 3 > 0$ for all $k$.

$2: \begin{cases} 1: \frac{dy}{dx} = 0 \text{ and } y = 0 \\ 1: \text{answer and explanation} \end{cases}$

(c) $P(x) = -4 + 6(x+1) - \frac{9}{2}(x+1)^2$

$2: \begin{cases} 1: \text{quadratic and centered at } x = -1 \\ 1: \text{coefficients} \end{cases}$

(d) $f(-1) = -4$

$f\left(-\frac{1}{2}\right) \approx -4 + \frac{1}{2}(6) = -1$

$f(0) \approx -1 + \frac{1}{2}\left(\frac{5}{4} + 2\right) = \frac{5}{8}$

$2: \begin{cases} 1: \text{Euler's method with 2 steps} \\ 1: \text{Euler's approximation to } f(0) \end{cases}$