Question 5

Consider the differential equation $\frac{dy}{dx} = 3x + 2y + 1$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

$$\frac{d}{dx}\left[3x+2y+1\right] = 3 + 2\frac{dy}{dx} = 3 + 2(3x+2y+1) = 3 + 6x + 4y + 2 = \boxed{6x+4y+5}$$

(c) Let $y = f(x)$ be a particular solution to the differential equation with the initial condition $f(0) = -2$. Use Euler's method, starting at $x = 0$ with a step size of $\frac{1}{2}$, to approximate $f(1)$. Show the work that leads to your answer.

x	y	$\frac{dy}{dx}$	Δx	$\Delta y = \frac{dy}{dx}\cdot\Delta x$	$x+\Delta x$	$y+\Delta y$
0	-2	-3	$1/2$	$-3/2$	$1/2$	$-7/2$
$1/2$	$-7/2$	$\frac{3}{2} - 14/2 + 2/2 = -9/2$	$1/2$	$-9/4$	1 $f(1) \approx$	$\boxed{-23/4}$

(d) Let $y = g(x)$ be another solution to the differential equation with the initial condition $g(0) = k$, where k is a constant. Euler's method, starting at $x = 0$ with a step size of 1, gives the approximation $g(1) \approx 0$. Find the value of k.

x	y	$\frac{dy}{dx}$	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	$x+\Delta x$	$y+\Delta y$
0	k	$2k+1$	1	$2k+1$	1	$0 = k+2k+1$ $0 = 3k+1$

$\boxed{k = -1/3}$

(b) Find the values of the constants m, b, and r for which $y = mx + b + e^{rx}$ is a solution to the differential equation.

$y = mx + b + e^{rx}$

$\frac{dy}{dx} = m + re^{rx} = 0x + m + re^{rx}$

$$\frac{dy}{dx} = 3x + 2y + 1$$
$$= 3x + 2(mx + b + e^{rx}) + 1$$
$$= 3x + 2mx + 2b + 2e^{rx} + 1$$
$$= (3+2m)x + 2b + 1 + 2e^{rx}$$

$3 + 2m = 0$

$\boxed{m = -3/2}$

$m = 2b + 1$

$-3/2 = 2b + 1$

$-5/2 = 2b$

$\boxed{b = -5/4}$

$2e^{rx} = re^{rx}$

$\therefore \boxed{r = 2}$

if $r = 0$ however:

$0x + m + 0 = (3+2m)x + 2b + 1 + 2 = (3+2m)x + 2b + 3$

$3 + 2m = 0$

$\boxed{m = -3/2}$

$m = 2b + 3$

$-3/2 = 2b + 3$

$-9/2 = 2b$

$\boxed{b = -9/4}$

$\boxed{r = 0}$

(a) $\dfrac{d^2y}{dx^2} = 3 + 2\dfrac{dy}{dx} = 3 + 2(3x + 2y + 1) = 6x + 4y + 5$

$2 : \begin{cases} 1 : 3 + 2\dfrac{dy}{dx} \\ 1 : \text{answer} \end{cases}$

(b) If $y = mx + b + e^{rx}$ is a solution, then
$m + re^{rx} = 3x + 2\left(mx + b + e^{rx}\right) + 1.$

If $r \neq 0$: $m = 2b + 1,\ r = 2,\ 0 = 3 + 2m,$
so $m = -\dfrac{3}{2},\ r = 2,$ and $b = -\dfrac{5}{4}.$

OR

If $r = 0$: $m = 2b + 3,\ r = 0,\ 0 = 3 + 2m,$
so $m = -\dfrac{3}{2},\ r = 0,\ b = -\dfrac{9}{4}.$

$3 : \begin{cases} 1 : \dfrac{dy}{dx} = m + re^{rx} \\ 1 : \text{value for } r \\ 1 : \text{values for } m \text{ and } b \end{cases}$

(c) $f\left(\frac{1}{2}\right) \approx f(0) + f'(0)\cdot\frac{1}{2} = -2 + (-3)\cdot\frac{1}{2} = -\frac{7}{2}$

$f'\left(\frac{1}{2}\right) \approx 3\left(\frac{1}{2}\right) + 2\left(-\frac{7}{2}\right) + 1 = -\frac{9}{2}$

$f(1) \approx f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\cdot\frac{1}{2} = -\frac{7}{2} + \left(-\frac{9}{2}\right)\cdot\frac{1}{2} = -\frac{23}{4}$

$2 : \begin{cases} 1 : \text{Euler's method with 2 steps} \\ 1 : \text{Euler's approximation for } f(1) \end{cases}$

(d) $g'(0) = 3\cdot 0 + 2\cdot k + 1 = 2k + 1$
$g(1) \approx g(0) + g'(0)\cdot 1 = k + (2k + 1) = 3k + 1 = 0$
$k = -\dfrac{1}{3}$

$2 : \begin{cases} 1 : g(0) + g'(0)\cdot 1 \\ 1 : \text{value of } k \end{cases}$