

AP® CALCULUS BC  
2007 SCORING GUIDELINES (Form B)

Question 6

MacLaurin

Let  $f$  be the function given by  $f(x) = 6e^{-x/3}$  for all  $x$ .

- (a) Find the first four nonzero terms and the general term for the Taylor series for  $f$  about  $x = 0$ .

$$\begin{aligned}
 f(x) &= 6e^{-x/3} & f(0) &= 6 & f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \\
 f'(x) &= 6\left(-\frac{1}{3}\right)e^{-x/3} & f'(0) &= -\frac{6}{3} & \text{(a)} \\
 f''(x) &= 6\left(-\frac{1}{3}\right)\left(\frac{1}{3}\right)e^{-x/3} & f''(0) &= \frac{6}{3^2} \\
 f'''(x) &= 6\left(-\frac{1}{3}\right)^3 e^{-x/3} & f'''(0) &= -\frac{6}{3^3} \\
 e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} \\
 e^{-x/3} &= 1 + \left(-\frac{x}{3}\right) + \left(-\frac{x}{3}\right)^2 \frac{1}{2!} + \left(-\frac{x}{3}\right)^3 \frac{1}{3!} + \dots + \left(-\frac{x}{3}\right)^n \frac{1}{n!} \\
 &= 1 - \frac{x}{3} + \frac{x^2}{3^2 \cdot 2!} - \frac{x^3}{3^3 \cdot 3!} + \dots + \frac{(-1)^n x^n}{3^n \cdot n!}
 \end{aligned}$$

$$6e^{-x/3} = 6 - \frac{6x}{3} + \frac{6x^2}{3^2 \cdot 2!} - \frac{6x^3}{3^3 \cdot 3!} + \dots + \frac{(-1)^n 6x^n}{3^n \cdot n!}$$

- (b) Let  $g$  be the function given by  $g(x) = \int_0^x f(t) dt$ . Find the first four nonzero terms and the general term for the Taylor series for  $g$  about  $x = 0$ .

Note:  $g(x) \rightarrow$  the integral of  $f(x)$

$$\begin{aligned}
 g(x) &\leftarrow \int_0^x 6e^{-t/3} dt = 6x - \frac{6x^2}{3 \cdot 2} + \frac{6x^3}{3 \cdot 3^2 \cdot 2!} - \frac{6x^4}{4 \cdot 3^3 \cdot 3!} + \dots + \frac{(-1)^n 6x^{n+1}}{(n+1) \cdot 3^n \cdot n!} \\
 &= \frac{6x}{1 \cdot 3 \cdot 0!} - \frac{6x^2}{2 \cdot 3 \cdot 1!} + \frac{6x^3}{3 \cdot 3^2 \cdot 2!} - \frac{6x^4}{4 \cdot 3^3 \cdot 3!} + \dots + \frac{(-1)^n 6x^{n+1}}{(n+1) \cdot 3^n \cdot n!}
 \end{aligned}$$

- (c) The function  $h$  satisfies  $h(x) = kf'(ax)$  for all  $x$ , where  $a$  and  $k$  are constants. The Taylor series for  $h$  about  $x = 0$  is given by

$$h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots \quad \leftarrow e^x$$

Find the values of  $a$  and  $k$ .

$$\begin{aligned}
 f(x) &= 6e^{-x/3} \\
 f'(x) &= -\frac{6}{3}e^{-x/3} \\
 f'(ax) &= -\frac{6}{3}e^{-ax/3} \\
 kf'(ax) &= -\frac{6k}{3}e^{-ax/3} = 1e^{1x} \\
 h(x) & \\
 \frac{-6k}{3} &= 1 \Rightarrow -2k=1 \Rightarrow k=-\frac{1}{2} \\
 -\frac{a}{3} &= 1 \Rightarrow a=-3
 \end{aligned}$$

$f'(3)$  means

$f'(t)$  evaluated at 3

Similarly,

$f'(ax)$  means

$f'(t)$  evaluated at  $ax$

$$(a) \quad f(x) = 6 \left[ 1 - \frac{x}{3} + \frac{x^2}{2!3^2} - \frac{x^3}{3!3^3} + \cdots + \frac{(-1)^n x^n}{n!3^n} + \cdots \right]$$

$$= 6 - 2x + \frac{x^2}{3} - \frac{x^3}{27} + \cdots + \frac{6(-1)^n x^n}{n!3^n} + \cdots$$

3 :  $\begin{cases} 1 : \text{two of } 6, -2x, \frac{x^2}{3}, -\frac{x^3}{27} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \\ \langle -1 \rangle \text{ missing factor of 6} \end{cases}$

$$(b) \quad g(0) = 0 \text{ and } g'(x) = f(x), \text{ so}$$

$$g(x) = 6 \left[ x - \frac{x^2}{6} + \frac{x^3}{3!3^2} - \frac{x^4}{4!3^3} + \cdots + \frac{(-1)^n x^{n+1}}{(n+1)!3^n} + \cdots \right]$$

$$= 6x - x^2 + \frac{x^3}{9} - \frac{x^4}{4(27)} + \cdots + \frac{6(-1)^n x^{n+1}}{(n+1)!3^n} + \cdots$$

3 :  $\begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \\ \langle -1 \rangle \text{ missing factor of 6} \end{cases}$

$$(c) \quad f'(x) = -2e^{-x/3}, \text{ so } h(x) = -2k e^{-ax/3}$$

$$h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = e^x$$

$$-2k e^{-ax/3} = e^x$$

$$\frac{-a}{3} = 1 \text{ and } -2k = 1$$

$$a = -3 \text{ and } k = -\frac{1}{2}$$

OR

$$f'(x) = -2 + \frac{2}{3}x + \cdots, \text{ so}$$

$$h(x) = kf'(ax) = -2k + \frac{2}{3}akx + \cdots$$

$$h(x) = 1 + x + \cdots$$

$$-2k = 1 \text{ and } \frac{2}{3}ak = 1$$

$$k = -\frac{1}{2} \text{ and } a = -3$$

3 :  $\begin{cases} 1 : \text{computes } kf'(ax) \\ 1 : \text{recognizes } h(x) = e^x, \\ \text{or} \\ \text{equates 2 series for } h(x) \\ 1 : \text{values for } a \text{ and } k \end{cases}$