Question 4

Let f be the function defined for $x > 0$, with $f(e) = 2$ and f', the first derivative of f, given by $f'(x) = x^2 \ln x$.

(a) Write an equation for the line tangent to the graph of f at the point $(e, 2)$.

(b) Is the graph of f concave up or concave down on the interval $1 < x < 3$? Give a reason for your answer.

(c) Use antidifferentiation to find $f(x)$.

a)

$y - y_1 = m(x - x_1)$

$f'(e) = e^2 \ln e = e^2$

$\boxed{y - 2 = e^2(x - e)}$

b) $f''(x) = x^2 \frac{1}{x} + 2x \ln x$

$= x + 2x \ln x = 0$ when

$x(1 + 2\ln x) = 0$

$x = 0 \quad 1 + 2\ln x = 0$

$2\ln x = -1$

$e^{\ln x} = e^{-1/2}$

$x = e^{-1/2} = \frac{1}{\sqrt{e}}$

$f''(x) = x + 2x\ln x > 0$ for $x > 1$

$\therefore f(x)$ is concave up on $1 < x < 3$

LIPET ← trig, exponential

log, inv. trig, polynomial

c)

$\int x^2 \ln x \, dx \quad u = \ln x \quad dv = x^2 dx$

$du = \frac{1}{x} dx \quad v = \frac{1}{3}x^3$

$\int u \, dv = uv - \int v \, du$

$= \frac{x^3 \ln x}{3} - \int \frac{x^3}{3x} dx = \frac{x^3 \ln x}{3} - \frac{1}{3}\int x^2 dx = \boxed{\frac{x^3 \ln x}{3} - \frac{1}{9}x^3 + C}$

(a) $f'(e) = e^2$

An equation for the line tangent to the graph of f at the point $(e, 2)$ is $y - 2 = e^2(x - e)$.

$2: \begin{cases} 1: f'(e) \\ 1: \text{equation of tangent line} \end{cases}$

(b) $f''(x) = x + 2x \ln x$.

For $1 < x < 3$, $x > 0$ and $\ln x > 0$, so $f''(x) > 0$. Thus, the graph of f is concave up on $(1, 3)$.

$3: \begin{cases} 2: f''(x) \\ 1: \text{answer with reason} \end{cases}$

(c) Since $f(x) = \int (x^2 \ln x) \, dx$, we consider integration by parts.

$u = \ln x \qquad dv = x^2 \, dx$

$du = \frac{1}{x} dx \qquad v = \int (x^2) \, dx = \frac{1}{3}x^3$

Therefore,

$$f(x) = \int (x^2 \ln x) \, dx = \frac{1}{3}x^3 \ln x - \int \left(\frac{1}{3}x^3 \cdot \frac{1}{x}\right) dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C.$$

Since $f(e) = 2$, $2 = \frac{e^3}{3} - \frac{e^3}{9} + C$ and $C = 2 - \frac{2}{9}e^3$.

Thus, $f(x) = \frac{x^3}{3} \ln x - \frac{1}{9}x^3 + 2 - \frac{2}{9}e^3$.

$4: \begin{cases} 2: \text{antiderivative} \\ 1: \text{uses } f(e) = 2 \\ 1: \text{answer} \end{cases}$