

AP[®] CALCULUS BC
2007 SCORING GUIDELINES

Question 6

Let f be the function given by $f(x) = e^{-x^2}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

$$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \dots +$$

$$1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots + \frac{(-1)^n x^{2n}}{n!}$$

0
 1
 2
 3

$f(x) = e^{-x^2}$	$f(0) = 1$
$f'(x) = -2xe^{-x^2}$	$f'(0) = 0$
$f''(x) = -2x(-2xe^{-x^2}) + e^{-x^2}(-2)$ $= 4x^2e^{-x^2} - 2e^{-x^2}$	$f''(0) = -2$
$f'''(x) = 4x^2(-2xe^{-x^2}) + e^{-x^2}(8x) - 2(-2xe^{-x^2})$ $= -8x^3e^{-x^2} + 8xe^{-x^2} + 4xe^{-x^2}$	$f'''(0) = 0$
⋮	$f^{(4)}(0) =$
⋮	$f^{(5)}(0) = 0$
⋮	$f^{(6)}(0) =$

In this way you would need $f^{(6)}(x)$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots + \frac{(-1)^n x^{2n}}{n!} + \dots$$

(b) Use your answer to part (a) to find $\lim_{x \rightarrow 0} \frac{1 - x^2 - f(x)}{x^4}$.

$$\lim_{x \rightarrow 0} \frac{1 - \cancel{x^2} - \left(1 - \cancel{x^2} + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots \right)}{x^4}$$

$$\lim_{x \rightarrow 0} \frac{-\frac{x^4}{2!x^4} + \frac{x^6}{3!x^4} - \frac{x^8}{4!x^4} + \dots}{x^4}$$

$$\lim_{x \rightarrow 0} \frac{-\frac{1}{2!} + \frac{x^2}{3!} - \frac{x^4}{4!} + \dots}{x^4} = \boxed{-\frac{1}{2}}$$

(c) Write the first four nonzero terms of the Taylor series for $\int_0^x e^{-t^2} dt$ about $x = 0$. Use the first two

terms of your answer to estimate $\int_0^{1/2} e^{-t^2} dt$.

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots + \frac{(-1)^n x^{2n}}{n!} + \dots$$

$$\int_0^x e^{-t^2} dt \approx \boxed{x - \frac{x^3}{3}} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!}$$

$$\int_0^{1/2} e^{-t^2} dt \approx \left(\frac{1}{2}\right) - \frac{\left(\frac{1}{2}\right)^3}{3} = \frac{1}{2} - \frac{1}{24} = \boxed{\frac{11}{24}}$$

(d) Explain why the estimate found in part (c) differs from the actual value of $\int_0^{1/2} e^{-t^2} dt$ by less than

$$\frac{1}{200}, \quad \frac{1}{2}, \quad \frac{1}{24}, \quad \frac{1}{320}$$

$$\int_0^x e^{-t^2} dt \approx x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!}$$

Alternating series converge when:

1. strictly alternating terms
2. $a_{n+1} < a_n$
3. $\lim_{n \rightarrow \infty} a_n = 0$

in addition, the magnitude of the error will be less than the magnitude of the first unused term i.e. $\frac{1}{320} < \frac{1}{200}$

$$(a) e^{-x^2} = 1 + \frac{(-x^2)}{1!} + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \dots + \frac{(-x^2)^n}{n!} + \dots$$

$$= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \dots + \frac{(-1)^n x^{2n}}{n!} + \dots$$

3 : $\begin{cases} 1 : \text{two of } 1, -x^2, \frac{x^4}{2}, -\frac{x^6}{6} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$

$$(b) \frac{1-x^2-f(x)}{x^4} = -\frac{1}{2} + \frac{x^2}{6} + \sum_{n=4}^{\infty} \frac{(-1)^{n+1} x^{2n-4}}{n!}$$

1 : answer

$$\text{Thus, } \lim_{x \rightarrow 0} \left(\frac{1-x^2-f(x)}{x^4} \right) = -\frac{1}{2}$$

$$(c) \int_0^x e^{-t^2} dt = \int_0^x \left(1 - t^2 + \frac{t^4}{2} - \frac{t^6}{6} + \dots + \frac{(-1)^n t^{2n}}{n!} + \dots \right) dt$$

$$= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \dots$$

3 : $\begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \\ 1 : \text{estimate} \end{cases}$

Using the first two terms of this series, we estimate that

$$\int_0^{1/2} e^{-t^2} dt \approx \frac{1}{2} - \left(\frac{1}{3}\right)\left(\frac{1}{8}\right) = \frac{11}{24}$$

$$(d) \left| \int_0^{1/2} e^{-t^2} dt - \frac{11}{24} \right| < \left(\frac{1}{2}\right)^5 \cdot \frac{1}{10} = \frac{1}{320} < \frac{1}{200}, \text{ since}$$

2 : $\begin{cases} 1 : \text{uses the third term as the error bound} \\ 1 : \text{explanation} \end{cases}$

$$\int_0^{1/2} e^{-t^2} dt = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)^{2n+1}}{n!(2n+1)}, \text{ which is an alternating series with individual terms that decrease in absolute value to 0.}$$

