$f(0)$ $f''(0)$

$f'(0)$ $f'''(0)$

6. Let f be the function given by $f(x) = \dfrac{2x}{1+x^2}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots + (-1)^n x^n$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots (-1)^n x^{2n}$$

$$\boxed{\frac{2x}{1+x^2} = \underset{0}{2x} - \underset{1}{2x^3} + \underset{2}{2x^5} - \underset{3}{2x^7} + \cdots + (-1)^n 2x^{2n+1}}$$

(b) Does the series found in part (a), when evaluated at $x = 1$, converge to $f(1)$? Explain why or why not.

$$a_n = (-1)^n 2(1)^{2n+1} = (-1)^n \cdot 2$$

$$\frac{d}{dx}\left[\ln(1+x^2)\right] = \frac{1}{1+x^2} \cdot 2x$$

$\lim\limits_{n\to\infty} (-1)^n \cdot 2 \neq 0$ therefore the series when evaluated at $x=1$ does not converge to $f(1)$.

(c) The derivative of $\ln(1+x^2)$ is $\dfrac{2x}{1+x^2}$. Write the first four nonzero terms of the Taylor series for $\ln(1+x^2)$ about $x = 0$.

$$\underset{0}{\frac{x^2}{1}} - \underset{1}{\frac{x^4}{2}} + \underset{2}{\frac{x^6}{3}} - \underset{3}{\frac{x^8}{4}} + \cdots + \frac{(-1)^n \cancel{2} x^{2n+2}}{2n+2} \quad \frac{(-1)^n x^{2n+2}}{n+1} \quad \cancel{2}(n+1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

$$\boxed{\ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4}}$$

estimate

actual

$$\ln\left(1+\tfrac{1}{4}\right) = \ln\left(1+\left(\tfrac{1}{2}\right)^2\right)$$

(d) Use the series found in part (c) to find a rational number A such that $\left|A - \ln\left(\frac{5}{4}\right)\right| < \frac{1}{100}$. Justify your answer.

Alternating Series Test

1. strictly alternating
2. $a_{n+1} < a_n$

$\frac{1}{4} > \frac{1}{32} > \boxed{\frac{1}{192}}$

3. $\lim\limits_{n\to\infty} a_n = 0$

if all are true, the alternating series converges

then the magnitude of the error will be less than the magnitude of the first unused term.

$$A = \frac{1}{4} - \frac{1}{32} = \frac{8}{32} - \frac{1}{32} = \boxed{\frac{7}{32}}$$

(a) $\dfrac{1}{1-u} = 1 + u + u^2 + \cdots + u^n + \cdots$

$\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + \left(-x^2\right)^n + \cdots$

$\dfrac{2x}{1+x^2} = 2x - 2x^3 + 2x^5 - 2x^7 + \cdots + (-1)^n 2x^{2n+1} + \cdots$

$3: \begin{cases} 1: \text{two of the first four terms} \\ 1: \text{remaining terms} \\ 1: \text{general term} \end{cases}$

(b) No, the series does not converge when $x = 1$ because when $x = 1$, the terms of the series do not converge to 0.

1 : answer with reason

(c) $\ln\left(1+x^2\right) = \displaystyle\int_0^x \frac{2t}{1+t^2}\,dt$

$= \displaystyle\int_0^x \left(2t - 2t^3 + 2t^5 - 2t^7 + \cdots\right) dt$

$= x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 + \cdots$

$2: \begin{cases} 1: \text{two of the first four terms} \\ 1: \text{remaining terms} \end{cases}$

(d) $\ln\left(\frac{5}{4}\right) = \ln\left(1+\frac{1}{4}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{1}{2}\right)^4 + \frac{1}{3}\left(\frac{1}{2}\right)^6 - \frac{1}{4}\left(\frac{1}{2}\right)^8 + \cdots$

Let $A = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^4 = \frac{7}{32}$.

Since the series is a converging alternating series and the absolute values of the individual terms decrease to 0,

$\left|A - \ln\left(\frac{5}{4}\right)\right| < \left|\frac{1}{3}\left(\frac{1}{2}\right)^6\right| = \frac{1}{3}\cdot\frac{1}{64} < \frac{1}{100}.$

$3: \begin{cases} 1: \text{uses } x = \frac{1}{2} \\ 1: \text{value of } A \\ 1: \text{justification} \end{cases}$