

**AP[®] CALCULUS BC
2009 SCORING GUIDELINES (Form B)**

Question 6

The function f is defined by the power series

$$f(x) = 1 + (x+1) + (x+1)^2 + \dots + (x+1)^n + \dots = \sum_{n=0}^{\infty} (x+1)^n$$

for all real numbers x for which the series converges.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots + x^n + \dots \quad \text{MacLaurin series about } x=0$$

then $1 + (x+1) + (x+1)^2 + (x+1)^3 + \dots + (x+1)^n + \dots = \frac{1}{1-(x+1)} = -\frac{1}{x}$

Geometric Series where $a=1$ $r=x+1$ Taylor series about $x=-1$

- (a) Find the interval of convergence of the power series for f . Justify your answer.

Geometric series converges when $|r| < 1$

$$|x+1| < 1$$

$$-1 < x+1 < 1$$

$$\boxed{-2 < x < 0}$$

$$f(x) = 1 + (x+1) + (x+1)^2 + (x+1)^3 + \dots + (x+1)^n$$

$$= \sum_{n=0}^{\infty} (x+1)^n$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x+1)^{n+1}}{(x+1)^n} \right| = |x+1|$$

If $|x+1| < 1$ then $\sum a_n$ converges

If $|x+1| > 1$ then $\sum a_n$ diverges

If $|x+1| = 1$ then inconclusive

So set $|x+1| < 1$ for $x = -2 \sum_{n=0}^{\infty} (-1)^n$

for $x = 0 \sum_{n=0}^{\infty} 1^n$ diverges because $\lim_{n \rightarrow \infty} (-1)^n \neq 0$

for $x = -2 \sum_{n=0}^{\infty} 1^n$ diverges because $\lim_{n \rightarrow \infty} 1^n \neq 0$

- (b) The power series above is the Taylor series for f about $x = -1$. Find the sum of the series for f .

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots + x^n + \dots \quad \text{MacLaurin Series centered at } 0.$$

$$\left(-\frac{1}{x} \right) = 1 + (x+1) + (x+1)^2 + (x+1)^3 + \dots + (x+1)^n + \dots \quad \text{Taylor series centered at } x = -1$$

Geometric series with $a=1$ and $r=x+1$

converges to $\frac{a}{1-r} = \frac{1}{1-(x+1)} = -\frac{1}{x}$ for $-2 < x < 0$

- (c) Let g be the function defined by $g(x) = \int_{-1}^x f(t) dt$. Find the value of $g\left(-\frac{1}{2}\right)$, if it exists, or explain why $g\left(-\frac{1}{2}\right)$ cannot be determined.

$$g\left(-\frac{1}{2}\right) = \int_{-1}^{-1/2} f(x) dx = - \int_{-1}^{-1/2} \frac{1}{x} dx$$

$$= - \left[\ln|x| \right]_{-1}^{-1/2}$$

$$= \left[\ln\left|\frac{1}{x}\right| \right]_{-1}^{-1/2}$$

$$= \ln 2 - \cancel{\ln 1}^0$$

$$= \ln 2$$

- (d) Let h be the function defined by $h(x) = f(x^2 - 1)$. Find the first three nonzero terms and the general term of the Taylor series for h about $x = 0$, and find the value of $h\left(\frac{1}{2}\right)$.

$$f(x) = 1 + (x+1) + (x+1)^2 + (x+1)^3 + \dots + (x+1)^n + \dots = -\frac{1}{x}$$

$$h(x) = f(x^2 - 1) = -\frac{1}{(x^2 - 1)} \quad h\left(\frac{1}{2}\right) = -\frac{1}{\left(\frac{1}{4} - 1\right)} = \frac{-1}{\frac{1}{4} - 1} = \frac{-1}{-\frac{3}{4}} = \boxed{\frac{4}{3}}$$

$$h(x) = 1 + (x-1+1) + (x-1+1)^2 + \dots$$

$$= 1 + \underset{0}{x} + \underset{1}{x^2} + \underset{2}{x^4} + \dots + \underset{n}{x^{2n}}$$

Scoring Guidelines

- (a) The power series is geometric with ratio $(x+1)$.
 The series converges if and only if $|x+1| < 1$.
 Therefore, the interval of convergence is $-2 < x < 0$.

OR

$$\lim_{n \rightarrow \infty} \left| \frac{(x+1)^{n+1}}{(x+1)^n} \right| = |x+1| < 1 \text{ when } -2 < x < 0$$

At $x = -2$, the series is $\sum_{n=0}^{\infty} (-1)^n$, which diverges since the terms do not converge to 0. At $x = 0$, the series is $\sum_{n=0}^{\infty} 1$, which similarly diverges. Therefore, the interval of convergence is $-2 < x < 0$.

3 : $\begin{cases} 1 : \text{identifies as geometric} \\ 1 : |x+1| < 1 \\ 1 : \text{interval of convergence} \end{cases}$

OR

3 : $\begin{cases} 1 : \text{sets up limit of ratio} \\ 1 : \text{radius of convergence} \\ 1 : \text{interval of convergence} \end{cases}$

- (b) Since the series is geometric,

$$f(x) = \sum_{n=0}^{\infty} (x+1)^n = \frac{1}{1-(x+1)} = -\frac{1}{x} \text{ for } -2 < x < 0.$$

1 : answer

$$(c) g\left(-\frac{1}{2}\right) = \int_{-1}^{-\frac{1}{2}} -\frac{1}{x} dx = -\ln|x| \Big|_{x=-1}^{x=-\frac{1}{2}} = \ln 2$$

2 : $\begin{cases} 1 : \text{antiderivative} \\ 1 : \text{value} \end{cases}$

$$(d) h(x) = f(x^2 - 1) = 1 + x^2 + x^4 + \dots + x^{2n} + \dots$$

$$h\left(\frac{1}{2}\right) = f\left(-\frac{3}{4}\right) = \frac{4}{3}$$

3 : $\begin{cases} 1 : \text{first three terms} \\ 1 : \text{general term} \\ 1 : \text{value of } h\left(\frac{1}{2}\right) \end{cases}$