AP® CALCULUS BC
2009 SCORING GUIDELINES

Question 4

Consider the differential equation $\frac{dy}{dx} = 6x^2 - x^2y$. Let $y = f(x)$ be a particular solution to this differential equation with the initial condition $f(-1) = 2$.

(a) Use Euler's method with two steps of equal size, starting at $x = -1$, to approximate $f(0)$. Show the work that leads to your answer.

a)

x	y	$\frac{dy}{dx}$	Δx	$\Delta y = \frac{dy}{dx}\Delta x$	$x+\Delta x$	$y+\Delta y$
-1	2	4	$.5$	2	$-.5$	4
$-.5$	4	$\frac{3}{2}-1=\frac{1}{2}$	$.5$	$1/4$	0	$4\frac{1}{4} = \boxed{17/4}$

(b) At the point $(-1, 2)$, the value of $\frac{d^2y}{dx^2}$ is -12. Find the second-degree Taylor polynomial for f about $x = -1$.

$$T_2(x) = f(-1) + f'(-1)(x+1) + \frac{f''(-1)(x+1)}{2!}$$

$$= 2 + 4(x+1) - \frac{12(x+1)^2}{2}$$

$$= \boxed{2 + 4(x+1) - 6(x+1)^2}$$

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(-1) = 2$.

$$\frac{dy}{dx} = 6x^2 - x^2y = x^2(6-y)$$

$$\int \frac{1}{6-y}\frac{dy}{\cancel{dx}}\cancel{dx} = \int x^2\,dx$$

$u = 6-y$
$du = -dy$
$-du = dy$

$$\int \frac{1}{6-y}\,dy = \int x^2\,dx$$

$$-\int \frac{du}{u} = \int x^2\,dx$$

$$-\ln|6-y| = \frac{x^3}{3} + C$$

$$e^{\ln|6-y|} = e^{-\frac{x^3}{3} + C}$$

$$|6-y| = e^C e^{-x^3/3}$$

$$6-y = \pm e^C e^{-x^3/3}$$

$$6-2 = \pm e^C e^{-(-1)^3/3}$$

$$(e^{-1/3})4 = \boxed{\pm e^C} e^{1/3}(e^{-1/3})$$

$$\pm e^C = 4e^{-1/3}$$

$$6-y = 4e^{-1/3}e^{-x^3/3}$$

$$y = 6 - 4e^{-1/3}e^{-x^3/3}$$

$$\boxed{y = 6 - 4e^{\frac{-(1+x^3)}{3}}}$$

(a) $f\left(-\frac{1}{2}\right) \approx f(-1) + \left(\left.\frac{dy}{dx}\right|_{(-1,\,2)}\right) \cdot \Delta x$

$= 2 + 4 \cdot \frac{1}{2} = 4$

$f(0) \approx f\left(-\frac{1}{2}\right) + \left(\left.\frac{dy}{dx}\right|_{\left(-\frac{1}{2},\,4\right)}\right) \cdot \Delta x$

$\approx 4 + \frac{1}{2} \cdot \frac{1}{2} = \frac{17}{4}$

$2 : \begin{cases} 1 : \text{Euler's method with two steps} \\ 1 : \text{answer} \end{cases}$

(b) $P_2(x) = 2 + 4(x+1) - 6(x+1)^2$

1 : answer

(c) $\frac{dy}{dx} = x^2(6 - y)$

$\int \frac{1}{6-y}\, dy = \int x^2\, dx$

$-\ln|6 - y| = \frac{1}{3}x^3 + C$

$-\ln 4 = -\frac{1}{3} + C$

$C = \frac{1}{3} - \ln 4$

$\ln|6 - y| = -\frac{1}{3}x^3 - \left(\frac{1}{3} - \ln 4\right)$

$|6 - y| = 4e^{-\frac{1}{3}\left(x^3+1\right)}$

$y = 6 - 4e^{-\frac{1}{3}\left(x^3+1\right)}$

$6 : \begin{cases} 1 : \text{separation of variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $3/6$ [1-2-0-0-0] if no constant of integration

Note: $0/6$ if no separation of variables