6. The Maclaurin series for e^x is $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots$. The continuous function f is defined by $f(x) = \frac{e^{(x-1)^2} - 1}{(x-1)^2}$ for $x \neq 1$ and $f(1) = 1$. The function f has derivatives of all orders at $x = 1$.

(a) Write the first four nonzero terms and the general term of the Taylor series for $e^{(x-1)^2}$ about $x = 1$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots \frac{x^{2n}}{n!}$$

$$\boxed{e^{(x-1)^2} = 1 + (x-1)^2 + \frac{(x-1)^4}{2!} + \frac{(x-1)^6}{3!} + \cdots \frac{(x-1)^{2n}}{n!}}$$

(b) Use the Taylor series found in part (a) to write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

$$e^{(x-1)^2} - 1 = 1 + (x-1)^2 + \frac{(x-1)^4}{2!} + \frac{(x-1)^6}{3!} + \cdots \frac{(x-1)^{2n}}{n!} \cdots - 1$$

$$= (x-1)^2 + \frac{(x-1)^4}{2!} + \frac{(x-1)^6}{3!} + \cdots$$

$$\frac{e^{(x-1)^2} - 1}{(x-1)^2} = \frac{(x-1)^2}{(x-1)^2} + \frac{(x-1)^4}{2!\,(x-1)^2} + \frac{(x-1)^6}{3!\,(x-1)^2} + \cdots$$

$$= \boxed{1 + \frac{(x-1)^2}{2!} + \frac{(x-1)^4}{3!} + \frac{(x-1)^6}{4!} + \cdots + \frac{(x-1)^{2n}}{(n+1)!}}$$

0 1 2 3

(c) Use the ratio test to find the interval of convergence for the Taylor series found in part (b).

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-1)^{2n+2}}{(n+2)!}\right| \cdot \left|\frac{(n+1)!}{(x-1)^{2n}}\right| = \lim_{n\to\infty}\frac{(n+1)!}{(n+2)!} \cdot \left|\frac{(x-1)^{2n+2}}{(x-1)^{2n}}\right|$$

$$= \lim_{n\to\infty}\frac{1}{n+2}\left|(x-1)^2\right|$$

$$= \lim_{n\to\infty}\frac{(x-1)^2}{n+2} = 0$$

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$ then $\sum a_n$ converges

for any x value

<u>converges for all x</u>

(d) Use the Taylor series for f about $x = 1$ to determine whether the graph of f has any points of inflection.

$$f(x) = \boxed{1 + \frac{(x-1)^2}{2!} + \frac{(x-1)^4}{3!} + \frac{(x-1)^6}{4!} + \dots + \frac{(x-1)^{2n}}{(n+1)!}}$$

$$f'(x) = \frac{2(x-1)}{2!} + \frac{4(x-1)^3}{3!} + \frac{6(x-1)^5}{4!} + \dots \qquad \sum_{n=1}^{\infty} \frac{2n(2n-1)(x-1)^{2n-2}}{(n+1)!}$$

$$= (x-1)^1 + \frac{4(x-1)^3}{3!} + \frac{6(x-1)^5}{4!} + \dots$$

$$f''(x) = 1 + \frac{4\cdot 3(x-1)^2}{3!} + \frac{6\cdot 5(x-1)^4}{4!} = 0 \text{ when ?? NEVER}$$

no points of inflection

$$\frac{d}{dx}\left[\frac{(x-1)^{2n}}{(n+1)!}\right] = \frac{2n(x-1)^{2n-1}}{(n+1)!} \qquad \frac{d}{dx}\left[\frac{2n(x-1)^{2n-1}}{(n+1)!}\right] = \frac{2n(2n-1)(x-1)^{2n-2}}{(n+1)!}$$

(a) $1 + (x-1)^2 + \frac{(x-1)^4}{2} + \frac{(x-1)^6}{6} + \cdots + \frac{(x-1)^{2n}}{n!} + \cdots$

$2 : \begin{cases} 1 : \text{first four terms} \\ 1 : \text{general term} \end{cases}$

(b) $1 + \frac{(x-1)^2}{2} + \frac{(x-1)^4}{6} + \frac{(x-1)^6}{24} + \cdots + \frac{(x-1)^{2n}}{(n+1)!} + \cdots$

$2 : \begin{cases} 1 : \text{first four terms} \\ 1 : \text{general term} \end{cases}$

(c) $$\lim_{n\to\infty} \left| \frac{\frac{(x-1)^{2n+2}}{(n+2)!}}{\frac{(x-1)^{2n}}{(n+1)!}} \right| = \lim_{n\to\infty} \frac{(n+1)!}{(n+2)!}(x-1)^2 = \lim_{n\to\infty} \frac{(x-1)^2}{n+2} = 0$$

Therefore, the interval of convergence is $(-\infty, \infty)$.

$3 : \begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{answer} \end{cases}$

(d) $$f''(x) = 1 + \frac{4\cdot 3}{6}(x-1)^2 + \frac{6\cdot 5}{24}(x-1)^4 + \cdots + \frac{2n(2n-1)}{(n+1)!}(x-1)^{2n-2} + \cdots$$

Since every term of this series is nonnegative, $f''(x) \geq 0$ for all x. Therefore, the graph of f has no points of inflection.

$2 : \begin{cases} 1 : f''(x) \\ 1 : \text{answer} \end{cases}$