1.

(a) Given $5x^3+40=\int_c^x f(t)\,dt$.

(i) Find $f(x)$.

(ii) Find the value of c.

(b) If $F(x)=\int_x^3 \sqrt{1+t^{16}}\,dt$, find $F'(x)$.

(a)

$$5x^3+40=\int_c^x f(t)\,dt$$

$$\frac{d}{dx}\left[5x^3+40\right]=\frac{d}{dx}\left[\int_c^x f(t)\,dt\right]$$

(i) $15x^2 = f(x)$

$$\int_c^x 15t^2\,dt=\left[5t^3\right]_c^x$$

$$=5x^3-5c^3$$

but $\int_c^x f(t)\,dt = 5x^3-40$,

so $\cancel{5x^3}+5c^3=\cancel{5x^3}-40$

$5c^3=40$

$c^3=8$

(ii) $c=2$

(b) $F(x)=\int_x^3\sqrt{1+t^{16}}$

$=-\int_3^x\sqrt{1+t^{16}}$

$F'(x)=-\sqrt{1+t^{16}}$

2.

Let f be a continuous function that is defined for all real numbers x and that has the following properties.

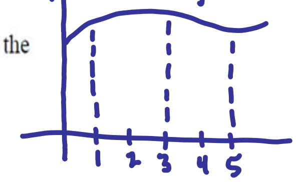
(i) $\int_1^3 f(x)\,dx=\frac{5}{2}$ (ii) $\int_1^5 f(x)\,dx=10$

(a) Find the average (mean) value of f over the closed interval $[1,3]$.

(b) Find the value of $\int_3^5 (2f(x)+6)\,dx$.

(c) Given that $f(x)=ax+b$, find the values of a and b.

$$\int_1^5 f(x)\,dx=\int_1^3 f(x)\,dx+\int_3^5 f(x)\,dx$$



(a) $\frac{1}{2}\int_1^3 f(x)\,dx=\frac{1}{2}\cdot\frac{5}{2}=\boxed{\frac{5}{4}}$

$$\int_3^5 2f(x)+6\,dx=2\int_3^5 f(x)\,dx+\underbrace{\int_3^5 6\,dx}_{12}$$

$$=2\left[\int_1^5 f(x)\,dx-\int_1^3 f(x)\,dx\right]+12$$

(b) $=2\left(10-\frac{5}{2}\right)+12=\boxed{27}$

$$\int_1^5 f(x)\,dx=\int_1^5 ax+b\,dx=\left[\frac{ax^2}{2}+bx\right]_1^5$$

$$=\left(\frac{25}{2}a+5b\right)-\left(\frac{1}{2}a+b\right)=12a+4b=10$$

$$\int_1^3 f(x)\,dx=\int_1^3 ax+b\,dx=\left[\frac{ax^2}{2}+bx\right]_1^3$$

$$=\left(\frac{9}{2}a+3b\right)-\left(\frac{1}{2}a+b\right)=4a+2b=\frac{5}{2}$$

so, $12a+4b=10$

$-2\left(4a+2b=\frac{5}{2}\right)$

(c) $4a=5$

$\boxed{a=5/4}$

$4\left(\frac{5}{4}\right)+2b=5/2$

$5+2b=5/2$

$2b=-5/2$

$\boxed{b=-5/4}$

3. $a(t) = 6t + 6 \quad v(0) = -9 \quad x(0) = -27$

A particle moves along the x-axis so that, at any time $t \geq 0$, its acceleration is given by $a(t) = 6t + 6$. At time $t = 0$, the velocity of the particle is -9, and its position is -27.

(a) Find $v(t)$, the velocity of the particle at any time $t \geq 0$.

(b) For what values of $t \geq 0$ is the particle moving to the right?

(c) Find $x(t)$, the position of the particle at any time $t \geq 0$.

(a) $v(t) = \int a(t)\,dt = \int 6t + 6\,dt = 3t^2 + 6t + C = \boxed{3t^2 + 6t - 9}$

$v(0) = 3(0)^2 + 6(0) + C = -9 \quad \therefore C = -9$

(b) $\boxed{\text{particle moves right when } v(t) > 0 \text{ i.e. when } t > 1}$

$3(t^2 + 2t - 3) = 0$

$3(t+3)(t-1) = 0$

$t = -3 \quad t = 1$

$v(t)$: $-$ $+$; 0, 1/2, 1, 2

(c) $x(t) = \int v(t)\,dt = \int 3t^2 + 6t - 9\,dt = t^3 + 3t^2 - 9t + C = \boxed{t^3 + 3t^2 - 9t - 27}$

$x(0) = (0)^3 + 3(0)^2 - 9(0) + C = -27 \quad \therefore C = -27$

4. $a(t) = \cos t \quad v(0) = 2 \quad x(0) = 5$

A particle moves along the x-axis with acceleration given by $a(t) = \cos t$ for $t \geq 0$. At $t = 0$, the velocity $v(t)$ of the particle is 2, and the position $x(t)$ is 5.

(a) Write an expression for the velocity $v(t)$ of the particle.

$v(t) = \int \cos t\,dt = \sin t + C = \boxed{\sin t + 2}$

$v(0) = \sin 0 + C = 2 \therefore C = 2$

(b) Write an expression for the position $x(t)$.

$x(t) = \int \sin t + 2\,dt = -\cos t + 2t + C = \boxed{-\cos t + 2t + 6}$

$x(0) = -\cos(0) + 2(0) + C = 5 \Rightarrow -1 + C = 5 \therefore C = 6$

(c) For what values of t is the particle moving to the right? Justify your answer.

$\boxed{\text{moves right when } v(t) > 0, \text{ i.e. } x > 0}$

$\sin t + 2 = 0$ never

always positive

(d) Find the total distance traveled by the particle from $t = 0$ to $t = \frac{\pi}{2}$.

(d) $\int_0^{\pi/2} \sin t + 2\,dt = \left[-\cos t + 2t\right]_0^{\pi/2}$

$= \left(-\cos\frac{\pi}{2} + 2\left(\frac{\pi}{2}\right)\right) - \left(-\cos 0 + 2(0)\right)$

$= 0 + \pi + 1 = \boxed{\pi + 1}$

because in part b we demonstrate that $v(t) > 0$ for all t, displacement equals total distance traveled

5. $a(t) = 6t - 18 \quad v(0) = 24 \quad x(1) = 20$

A particle moves along the x-axis so that its acceleration at any time t is given by $a(t) = 6t - 18$. At time $t = 0$ the velocity of the particle is $v(0) = 24$, and at time $t = 1$, its position is $x(1) = 20$.

(a) $v(t) = \int 6t - 18\,dt = 3t^2 - 18t + C = \boxed{3t^2 - 18t + 24}$

$v(0) = 3(0)^2 - 18(0) + C = 24 \quad \therefore C = 24$

(a) Write an expression for the velocity $v(t)$ of the particle at any time t.

(b) For what values of t is the particle at rest?

$3t^2 - 18t + 24 = 0 \Rightarrow 3(t^2 - 6t + 8) = 0$

$3(t-4)(t-2) = 0$

$t = 4 \quad t = 2$

(b) at rest when $v(t) = 0$, i.e. $\boxed{t = 2, t = 4}$

(c) Write an expression for the position $x(t)$ of the particle at any time t.

(d) Find the total distance traveled by the particle from $t = 1$ to $t = 3$.

(c) $x(t) = \int 3t^2 - 18t + 24\,dt = t^3 - 9t^2 + 24t + C = \boxed{t^3 - 9t^2 + 24t + 4}$

$x(1) = (1)^3 - 9(1)^2 + 24(1) + C = 20 \quad \therefore C = 4$

(d) $v(t)$: $+$ $-$; 1, 2, 3

Displacement would be $\int_1^3 v(t)\,dt$, but total distance traveled is $\int_1^2 v(t)\,dt + \left|\int_2^3 v(t)\,dt\right|$ since $v(t)<0$ for $2<t<3$.

$$\int_1^2 v(t)\,dt = x(2)-x(1)$$
$$= \left((2)^3-9(2)^2+24(2)+4\right)-\left(1^3-9(1)^2+24(1)+4\right)$$
$$= (8-36+48+4)-(1-9+24+4)$$
$$= 24-20=4$$

see above

$$\left|\int_2^3 v(t)\,dt\right| = |x(3)-x(2)|$$
$$= \left|\left((3)^3-9(3)^2+24(3)+4\right)-24\right|$$
$$= |(27-81+72+4)-24|$$
$$= |22-24| = |-2| = 2$$

so $\boxed{\int_1^2 v(t)\,dt + \left|\int_2^3 v(t)\,dt\right| = 4+2=6}$

6.

Let f be a differentiable function, defined for all real numbers x, with the following properties.

(i) $f'(x) = ax^2 + bx$

(ii) $f'(1) = 6$ and $f''(1) = 18$

(iii) $\int_1^2 f(x)\,dx = 18$

Find $f(x)$. Show your work.

$$f'(1) = a(1)^2+b(1) = a+b=6$$
$$f''(x) = 2ax+b$$
$$f''(1) = 2a(1)+b = 2a+b=18$$

$$2a+b=18$$
$$-1(a+b=6)$$
$$a = 12$$
$$b = -6$$

$$f(x) = \int f'(x)\,dx = \int 12x^2-6x\,dx = 4x^3-3x^2+C$$
$$\int_1^2 f(x)\,dx = \int_1^2 4x^3-3x^2+C\,dx = \left[x^4-x^3+Cx\right]_1^2$$
$$= (16-8+2C)-(1-1+C)$$
$$= 8+C = 18 \quad \therefore \quad C=10$$

and $\boxed{f(x) = 4x^3-3x^2+10}$

7.

$$f'(x) = \int f''(x)\,dx = \int 6x+8\,dx = 3x^2+8x+C = 3x^2+8x+3$$
$$f'(0) = 3(0)^2+8(0)+C = 3 \quad \therefore \quad C=3$$

Let f be a function such that $f''(x) = 6x+8$.

(a) Find $f(x)$ if the graph of f is tangent to the line $3x-y=2$ at the point $(0,-2)$.

Since $y=3x-2$ is tangent to $f(x)$ at $(0,-2)$, $f'(0)=3$ and $f(0)=-2$

(b) Find the average value of $f(x)$ on the closed interval $[-1,1]$.

(a)

$$f(x) = \int f'(x)\,dx = \int 3x^2+8x+3\,dx = x^3+4x^2+3x+C = \boxed{x^3+4x^2+3x-2}$$
$$f(0) = (0)^3+4(0)^2+3(0)+C = -2 \quad \therefore \quad C=-2$$

(b) $\frac{1}{2}\int_{-1}^{1} x^3+4x^2+3x-2\,dx = \frac{1}{2}\left[\frac{x^4}{4}+\frac{4x^3}{3}+\frac{3x^2}{2}-2x\right]_{-1}^{1}$

$= \frac{1}{2}\left[\left(\frac14+\frac43+\frac32-2\right)-\left(\frac14-\frac43+\frac32+2\right)\right]$

$= \frac{1}{2}\left[\frac83-4\right] = \frac12\left(-\frac43\right) = -\frac46 = \boxed{-\frac23}$

8.

$v(0)=0$ $\quad x(1)=3$ $\quad a(t)=12t^2-4$

A particle, initially at rest, moves along the x-axis so that its acceleration at any time $t \geq 0$ is given by $a(t)=12t^2-4$. The position of the particle when $t=1$ is $x(1)=3$.

(a) Find the values of t for which the particle is at rest.

when $v(t)=0$ $\quad v(t)=4t^3-4t=0$ when $4t(t^2-1)=0$ $\quad \boxed{\text{only } t=0,\ t=1 \text{ since } t\geq 0}$

(b) Write an expression for the position $x(t)$ of the particle at any time $t \geq 0$.

$x(t)=t^4-2t^2+4$

(c) Find the total distance traveled by the particle from $t=0$ to $t=2$.

$v(t)$: $-$ on $(0,1)$, $+$ on $(1,2)$; 0, 1, 2

$v(t)=\int 12t^2-4\,dt = 4t^3-4t+C = 4t^3-4t$

$v(0)=4(0)^3+4(0)+C=0 \quad \therefore C=0$

(b) $x(t)=\int 4t^3-4t\,dt = t^4-2t^2+C = \boxed{t^4-2t^2+4}$

$x(1)=1^4-2(1)^2+C=3 \quad \therefore C=4$

(c) Total Distance traveled $= \left|\int_0^1 v(t)\,dt\right| + \int_1^2 v(t)\,dt$ $\quad$ Since $v(t)<0$ on $(0,1)$ and $v(t)>0$ on $(1,2)$

$= |x(1)-x(0)| + x(2)-x(1)$

$= |3-4| + (16-8+4)-3$

$= |-1| + 9 = \boxed{10}$

9.

Let f be the function that is defined for all real numbers x and that has the following properties.

(i) $f''(x)=24x-18$

(ii) $f'(1)=-6$

(iii) $f(2)=0$

$f'(x)=\int f''(x)\,dx = \int 24x-18\,dx = 12x^2-18x+C = 12x^2-18x$

$f'(1)=12(1)^2-18(1)+C=-6 \quad \therefore C=0$

$f'(x)=12x^2-18x=0$ when $6x(2x-3)=0$

(a) i.e. $\boxed{x=0 \quad x=3/2}$

(a) Find each x such that the line tangent to the graph of f at $(x, f(x))$ is horizontal.

when $f'(x)=0$, i.e. $x=0$ $\quad x=3/2$

(b) Write an expression for $f(x)$.

$f(x)=4x^3-9x^2+4$

(c) Find the average value of f on the interval $1 \leq x \leq 3$.

(b) $f(x)=\int f'(x)\,dx = \int 12x^2-18x\,dx = 4x^3-9x^2+C = \boxed{4x^3-9x^2+4}$

$f(2)=4(2)^3-9(2)^2+C=0 \quad \therefore C=4$

(c) Avg value of $f(x)$ on $[1,3]$ $= \frac12\int_1^3 4x^3-9x^2+4\,dx = \frac12\left[x^4-3x^3+4x\right]_1^3 = \frac12\left[(81-81+12)-(1-3+4)\right]$

$= \frac12(12-2) = \frac12(10) = \boxed{5}$

(a) $x(t) = \int v(t)\,dt = \int 12t^2 - 36t + 15\,dt = 4t^3 - 18t^2 + 15t + C = \boxed{4t^3 - 18t^2 + 15t - 1}$

10. $x(1) = 4(1)^3 - 18(1)^2 + 15(1) + C = 0 \therefore C = -1$

A particle moves on the x-axis so that its velocity at any time $t \geq 0$ is given by $v(t) = 12t^2 - 36t + 15$. At $t = 1$, the particle is at the origin.

$v(t) = 12t^2 - 36t + 15$
$= 3(4t^2 - 12t + 5)$
$= 3(2t-1)(2t-5)$
$= 0$ when $\boxed{t = 1/2, t = 5/2}$ (b)

(a) Find the position $x(t)$ of the particle at any time $t \geq 0$.

$x(t) = 4t^3 - 18t^2 + 15t - 1$

(b) Find all values of t for which the particle is at rest.

when $v(t) = 0$, i.e. when $t = 1/2$, $t = 5/2$

(c) Find the maximum velocity of the particle for $0 \leq t \leq 2$.

test $v(t)$ at critical values and end points (note: $5/2 > 2$) i.e. $v(0) = 15$

(d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

$v(t)$: + on $(0, 1/2)$, − on $(1/2, 2)$

(c)

t	$v(t)$
0	15 MAX
1/2	$3-18+15=0$
2	-9

tot. distance $= \int_0^{1/2} v(t)\,dt + \left|\int_{1/2}^{2} v(t)\,dt\right|$

(d)

$$= x(\tfrac{1}{2}) - x(0) + |x(2) - x(\tfrac{1}{2})|$$
$$= \left(\tfrac{1}{2} - \tfrac{9}{2} + \tfrac{15}{2} - 1\right) - (-1) + \left|(32 - 72 + 30 - 1) - (\tfrac{5}{2})\right|$$
$$= \tfrac{5}{2} + 1 + \left|-11 - \tfrac{7}{2}\right| = \tfrac{7}{2} + \left|-\tfrac{22}{2} - \tfrac{5}{2}\right| = \tfrac{7}{2} + \tfrac{27}{2} = \tfrac{34}{2} = \boxed{17}$$

(a) $x(t) = \int v(t)\,dt = \int 3t^2 - 2t - 1\,dt = t^3 - t^2 - t + C = \boxed{t^3 - t^2 - t + 3}$

11. $x(2) = 8 - 4 - 2 + C = 5 \therefore C = 3$

A particle moves along the x-axis so that its velocity at any time $t \geq 0$ is given by $v(t) = 3t^2 - 2t - 1$. The position $x(t)$ is 5 for $t = 2$. $x(2) = 5$

(b) $\dfrac{x(3) - x(0)}{3} = v(t)$

$\dfrac{18 - 3}{3} = 3t^2 - 2t - 1$

$5 = 3t^2 - 2t - 1$

$3t^2 - 2t - 6 = 0$ when

$x = \dfrac{2 \pm \sqrt{4 - 4(3)(-6)}}{6}$

$= \dfrac{2 \pm \sqrt{76}}{6} = \dfrac{2 \pm 2\sqrt{19}}{6}$

$= \boxed{\dfrac{1 \pm \sqrt{19}}{3}}$

(a) Write a polynomial expression for the position of the particle at any time $t \geq 0$.

$x(t) = t^3 - t^2 - t + 3$

(b) For what values of t, $0 \leq t \leq 3$, is the particle's instantaneous velocity the same as its average velocity on the closed interval $[0, 3]$? $t = \dfrac{1 \pm \sqrt{19}}{3}$

(c) Find the total distance traveled by the particle from time $t = 0$ until time $t = 3$.

$v(t) = 0$ when $3t^2 - 2t - 1 = 0$ only use $t = 1$
$(3t+1)(t-1)$ since $t \geq 0$
$t = -1/3$ $t = 1$

$v(t)$: − on $(0, 1)$, + on $(1, 3)$

total distance $= \left|\int_0^1 v(t)\,dt\right| + \int_1^3 v(t)\,dt$

(c)

$$= |x(1) - x(0)| + x(3) - x(1)$$
$$= |2 - 3| + (27 - 9 - 3 + 3) - 2$$
$$= 1 + 18 - 2$$
$$= \boxed{17}$$