Given $f(x)=\frac{1}{x}+\ln x$, defined only on the closed interval $\frac{1}{e}\le x\le e$.

$f(1/e)=1/(1/e)+\ln(1/e)=e+\ln 1-\ln e=e-1\approx 1.718$ abs. max

$f(1)=1/1+\ln 1=1$ abs. min

(a) Showing your reasoning, determine the value of x at which f has its

$f(e)=1/e+\ln e=1/e+1\approx .5+1=1.5$

(i) absolute maximum,

(ii) absolute minimum.

(b) For what values of x is the curve concave up?

(c) On the coordinate axes provided, sketch the graph of f over the interval $\frac{1}{e}\le x\le e$.

a) abs max/min occur at critical values or endpoints

$f(x)=\frac{1}{x}+\ln x$

$f'(x)=-\frac{1}{x^2}+\frac{1}{x}\left(\frac{x}{x}\right)$

$=\frac{x-1}{x^2}$

critical value of $x=1$

(don't include $x=0$ b/c function not defined there)

x	$f(x)$
$1/e$	≈ 1.718 ← MAX
1	1 ← MIN
e	≈ 1.5

b) $f(x)$ concave up when $f''(x)>0$

$f'(x)=\frac{x-1}{x^2}$ $\quad f''(x)=\frac{x^2(1)-(x-1)(2x)}{x^4}$

$=\frac{x^2-2x^2+2x}{x^4}$

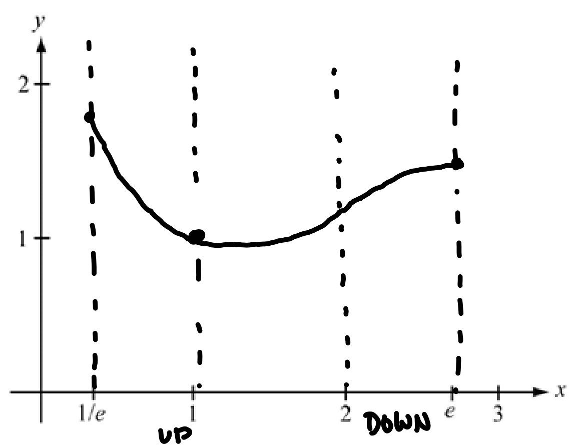
$=\frac{2x-x^2}{4}=\frac{x(2-x)}{x^4}$

$=\frac{2-x}{x^3}=0$ when $x=2$

$x\neq 0$

$f''(x)$: $1/e$ … $+$ … 1 … 2 … $-$ … 2.5 … e

$f(x)$ concave up on $(0,2)$



Consider the function f given by $f(x) = x^{\frac{4}{3}} + 4x^{\frac{1}{3}}$ on the interval $-8 \le x \le 8$.

(a) Find the coordinates of all points at which the tangent to the curve is a horizontal line.

(b) Find the coordinates of all points at which the tangent to the curve is a vertical line.

(c) Find the coordinates of all points at which the absolute maximum and absolute minimum occur.

(d) For what values of x is this function concave down?

(e) On the axes provided, sketch the graph of the function on this interval.

a) tan to curve is horizontal when $f'(x) = 0$, i.e. when $x = -1$

$f(x) = x^{4/3} + 4x^{1/3}$

$f'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-2/3}\left[x+1\right] = 0$ when $x = -1$ $(-1, -3)$

b) tan to curve is vertical line when $f'(x)$ DNE, assuming $f(x)$ exists at that point,

$f'(x)$ DNE when $x^{-2/3} = 0$ or $x = 0$. i.e., when $x = 0$ $(0,0)$

c) abs. max/min occur at critical values or end points

in this case, both $x = -1$ and $x = 0$ are critical points

$f(x) = x^{4/3} + 4x^{1/3}$

Abs min @ $(-1, -3)$

Abs max @ $(8, 24)$

x	$f(x)$
-8	$16-8=8$
-1	-3
0	0
8	$16+8=24$

d) $f(x)$ concave down when $f''(x) < 0$, i.e. on $(-\infty, 0) \cup (0, 2)$

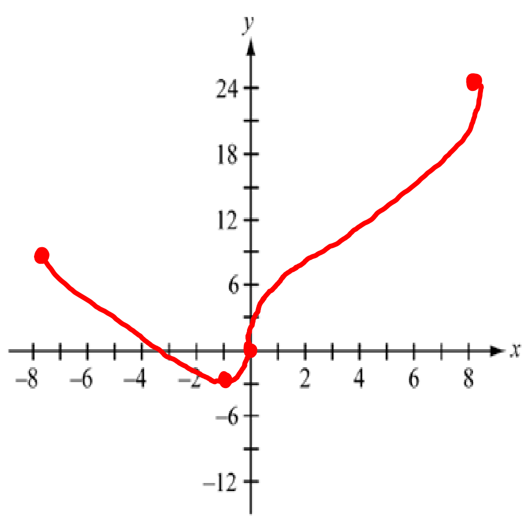
$f'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3}$

$f''(x) = \frac{4}{9}x^{-2/3} - \frac{8}{9}x^{-5/3} = 0$ when

$\frac{4}{9}x^{-5/3}\left[x-2\right] = 0$

$x \ne 0$ $x = 2$

$f''(x)$: $+$ on $(-8, 0)$, $-$ on $(0, 2)$, $+$ on $(2, 8)$



Consider $f(x) = \cos^2 x + 2\cos x$ over one complete period beginning with $x = 0$.

(a) Find all values of x in this period at which $f(x) = 0$.

(b) Find all values of x in this period at which the function has a minimum. Justify your answer.

(c) Over what intervals in this period is the curve concave up?

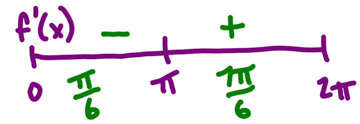
a) $f(x) = \cos^2 x + 2\cos x = 0$ when $\cos x = 0$ i.e. $x = \frac{\pi}{2}, \frac{3\pi}{2}$

b) $f(x)$ has a min when $f'(x)$ changes from $-$ to $+$

$f'(x) = -2\cos x \sin x - 2\sin x = 0$ when

$-2\sin x(\cos x + 2) = 0$

$\sin x = 0$, i.e. $x = 0, \pi, 2\pi$



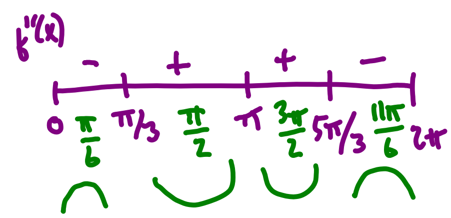
c) $f(x)$ concave up when $f''(x) > 0$, i.e. on $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$

$$\begin{aligned} f''(x) &= -2\cos x \cos x + 2\sin x \sin x - 2\cos x \\ &= -2[\cos^2 x - \sin^2 x] - 2\cos x \\ &= -2[\cos^2 x - (1 - \cos^2 x)] - 2\cos x \\ &= -2[2\cos^2 x - 1] - 2\cos x \\ &= -4\cos^2 x - 2\cos x + 2 \\ &= -2[2\cos^2 x + \cos x - 1] \\ &= -2(2\cos x - 1)(\cos x + 1) \\ &= 0 \text{ when } \pi/3, 5\pi/3, \pi \end{aligned}$$

$2\cos^2 x + \cos x - 1$

$2u^2 + u - 1$

$(2u - 1)(u + 1)$



Given the function defined by $y = x + \sin x$ for all x such that $-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$.

(a) Find the coordinates of <u>all</u> maximum and minimum points on the given interval. Justify your answers.

(b) Find the coordinates of <u>all</u> points of inflection on the given interval. Justify your answers.

(c) On the axes provided, sketch the graph of the function.

a) Abs. Max and Abs. min occur at critical values and end points

$y = x + \sin x$

$y' = 1 + \cos x$

$= 0$ when $x = \pi$

	x	$f(x)$
ABS MIN →	$-\pi/2$	$-\pi/2 - 1 \approx -2.5$
ABS	π	$\pi \approx 3.14$
MAX →	$3\pi/2$	$3\pi/2 - 1 \approx 3.5$

Rel. min/max occur at critical values

No rel. min/max f(x) is monotone increasing

$f'(x)$: $+$ $+$ on $-\pi/2,\ \pi/3,\ \pi,\ 4\pi/3,\ 3\pi/2$

b) Pts. of inflection occur when $f''(x)$ changes sign, i.e. @ $x = 0, x = \pi$

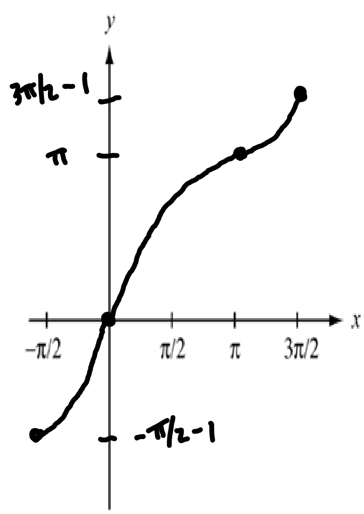
$y' = 1 + \cos x$

$y'' = -\sin x$

$= 0$ when $x = 0, \pi$

$+$ $-$ $+$ on $-\pi/2,\ -\pi/6,\ 0,\ \pi/2,\ \pi,\ \frac{7\pi}{6},\ 3\pi/2$

(c) On the axes provided, sketch the graph of the function.



Consider the function f defined by $f(x) = (x^2 - 1)^3$ for all real numbers x.

(a) For what values of x is the function increasing?

(b) Find the x- and y-coordinates of the relative maximum and minimum points. Justify your answer.

(c) For what values of x is the graph of f concave upward?

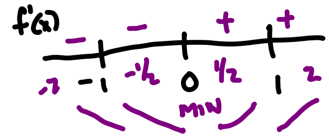
(d) Using the information found in parts (a), (b), and (c), sketch the graph of f on the axes provided.

a) $f(x)$ increases when $f'(x) > 0$, i.e. on $x > 0$

$f(x) = (x^2-1)^3$

$f'(x) = 3(x^2-1)^2(2x) = 6x(x^2-1)^2$

$= 0$ when $x = -1, 0, 1$



b) Rel max occurs when $f'(x)$ changes from $+$ to $-$, but this never happens

Rel min occurs when $f'(x)$ changes from $-$ to $+$, i.e. @ $x = 0$

c) $f(x)$ is concave up when $f''(x) > 0$

$f'(x) = 6x(x^2-1)^2 = 6x[x^4 - 2x^2 + 1] = 6x^5 - 12x^3 + 6x$

let $u = x^2$

$f''(x) = 30x^4 - 36x^2 + 6 = 6[5x^4 - 6x^2 + 1]$

$= 6[5u^2 - 6u + 1]$

$f''(x) = 6(5x^2-1)(x^2-1)$

$= 6(5u-1)(u-1)$

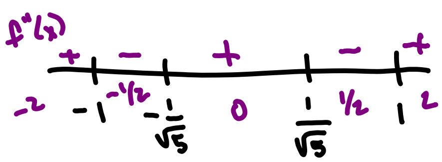
$= 0$ when $5u = 1$, $u = 1$

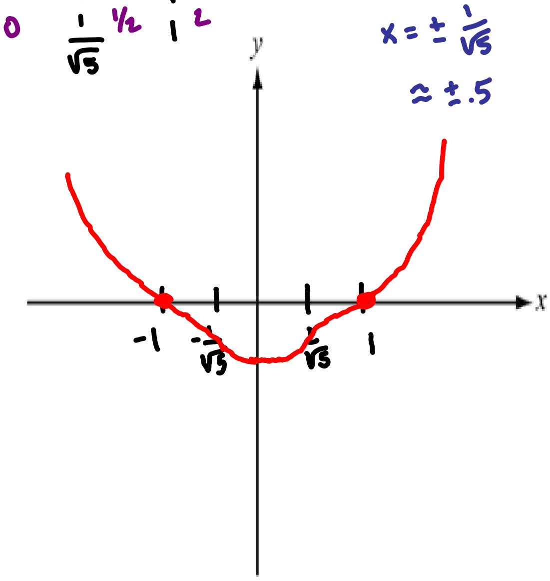
$5x^2 = 1$, $x^2 = 1$

$x^2 = 1/5$, $x = \pm 1$

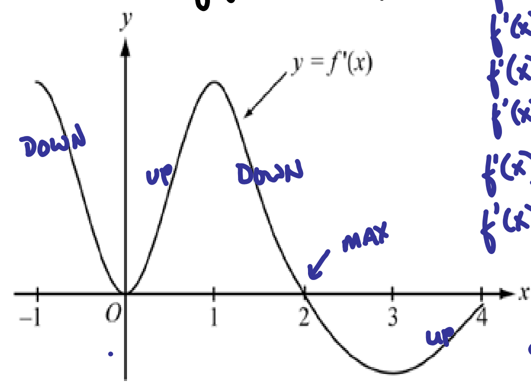
$x = \pm \frac{1}{\sqrt{5}}$

$\approx \pm .5$





Remember, this is the graph of $f'(x)$, not $f(x)$



a) $f'(x) > 0 \Rightarrow f(x)$ increasing
$f'(x) < 0 \Rightarrow f(x)$ decreasing
$f'(x)$ changes $+$ to $- \Rightarrow$ rel. max
$f'(x)$ changes $-$ to $+ \Rightarrow$ rel. min
$f'(x)$ increasing $\Rightarrow f''(x) > 0$
$f'(x)$ decreasing $\Rightarrow f''(x) < 0$
$f'(x)$ changes from increasing to decreasing or decreasing to increasing $\Rightarrow$ point of inflection

Note: This is the graph of the derivative of f, NOT the graph of f.

Let f be a function that has domain the closed interval $[-1,4]$ and range the closed interval $[-1,2]$. Let $f(-1) = -1$, $f(0) = 0$, and $f(4) = 1$. Also let f have the derivative function f' that is continuous and that has the graph shown in the figure above.

(a) Find all values of x for which f assumes a relative maximum. Justify your answer.

Rel. max when $f'(x)$ changes $+$ to $-$, i.e. @ $x = 2$

(b) Find all values of x for which f assumes its absolute minimum. Justify your answer.

Abs. min @ critical values or endpoints. $(-1,-1)$

abs. min.

x	$f(x)$
-1	-1
0	0
2	??
4	1

rel. max cannot be abs. min.

(c) Find the intervals on which f is concave downward.

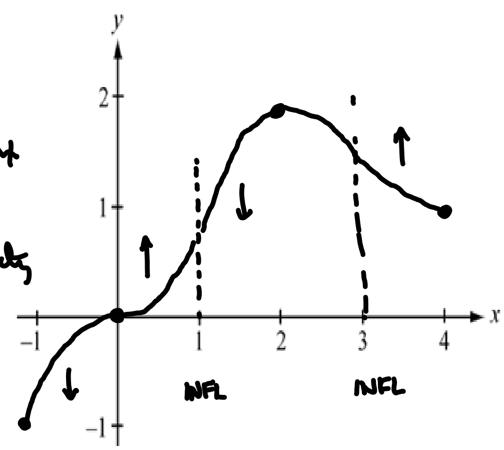
$f(x)$ concave down when $f'(x)$ decreasing, i.e. on $(-1,0) \cup (1,3)$

(d) Give all the values of x for which f has a point of inflection.

$f(x)$ has pt. of inflection when $f'(x)$ changes incr to decr or decr to incr. i.e. $x=0$, $x=1$, $x=3$

(e) On the axes provided, sketch the graph of f.

graphing
1. plot points
2. plot rel min/max
3. identify POI
4. graph concavity



Let f be the function defined by $f(x) = 12x^{\frac{2}{3}} - 4x$.

a) $f(x) = 12x^{2/3} - 4x$

$f'(x) = \frac{2}{3}12x^{-1/3} - 4$

$= \boxed{8x^{-1/3} - 4}$

$= 0$ when $4(2x^{-1/3} - 1) = 0$

$\frac{2}{x^{1/3}} = \frac{1}{1}$

$x^{1/3} = 2$

$x = 8$

$x \neq 0$

(a) Find the intervals on which f is increasing.

when $f'(x) > 0$, i.e. on $(0, 8)$

(b) Find the x- and y-coordinates of all relative maximum points.

when $f'(x)$ changes $+$ to $-$, i.e. @ $x = 8$ $(8, 16)$

(c) Find the x- and y-coordinates of all relative minimum points.

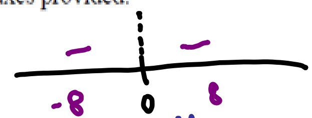
when $f'(x)$ changes $-$ to $+$, i.e. @ $x = 0$ $(0,0)$

(d) Find the intervals on which f is concave downward.

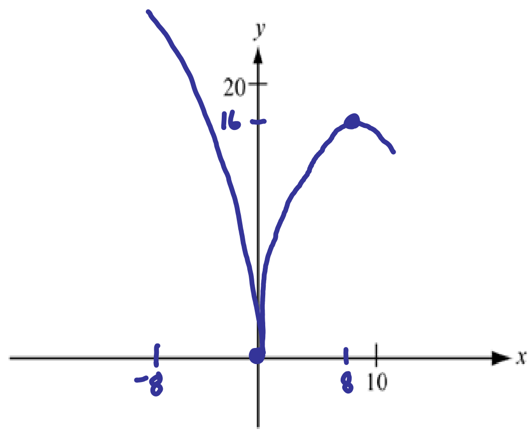
d) $f'(x) = 8x^{-1/3} - 4$

$f''(x) = -\frac{1}{3}8x^{-4/3}$ $x \neq 0$

(e) Using the information found in parts (a), (b), (c), and (d), sketch the graph of f on the axes provided.



no change in concavity



A function f is continuous on the closed interval $[-3, 3]$ such that $f(-3) = 4$ and $f(3) = 1$. The functions f' and f'' have the properties given in the table below.

x	$-3 < x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$1 < x < 3$
$f'(x)$	Positive INCR	MAX Fails to exist SHARP TURN	Negative DECR	0	Negative DECR
$f''(x)$	Positive UP	Fails to exist	Positive UP	INFL 0	Negative DOWN

x	$f(x)$
-3	4
-1	
1	
3	1

rel. and abs. max

abs. min

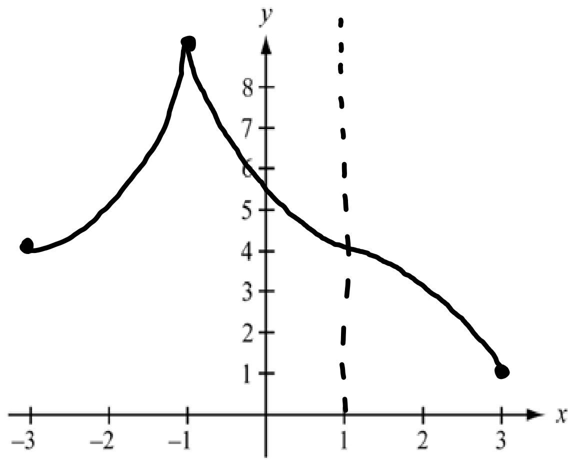
(a) What are the x-coordinates of all absolute maximum and absolute minimum points of f on the interval $[-3, 3]$? Justify your answer.

Abs max at critical values or end points   abs. max @ $x=-1$ abs. min @ $x=3$

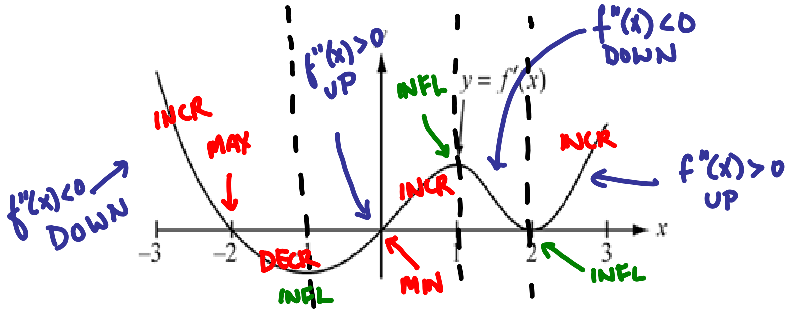
(b) What are the x-coordinates of all points of inflection of f on the interval $[-3, 3]$? Justify your answer.

Pts. of inflection when $f''(x)$ changes sign, i.e. @ $x=1$ only

(c) On the axes provided, sketch a graph that satisfies the given properties of f.



INFL



Note: This is the graph of the <u>derivative</u> of f, <u>not</u> the graph of f.

The figure above shows the graph of f', the derivative of a function f. The domain of the function f is the set of all x such that $-3 \le x \le 3$.

(a) For what values of x, $-3 < x < 3$, does f have a relative maximum? A relative minimum? Justify your answer.
Rel. max when $f'(x)$ changes + to −, i.e. @ $x = -2$
Rel. min when $f'(x)$ changes − to +, i.e. @ $x = 0$

(b) For what values of x is the graph of f concave up? Justify your answer.
when $f'(x)$ is increasing ($f''(x) > 0$) i.e. on $(-1, 1) \cup (2, 3)$

(c) Use the information found in parts (a) and (b) and the fact that $f(-3) = 0$ to sketch a possible graph of f on the axes provided below.