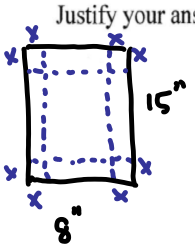
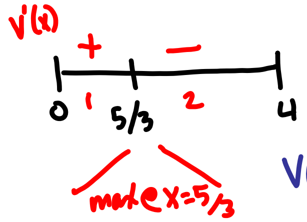
1. Find the maximum volume of a box that can be made by cutting out squares from the corners of an 8-inch by 15-inch rectangular sheet of cardboard and folding up the sides. Justify your answer.



$V = lwh$

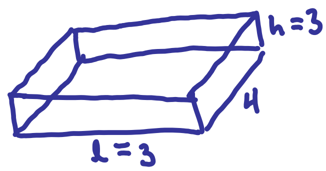
$V(x) = (15-2x)(8-2x)x$
$\quad = 4x^3 - 46x^2 + 120x$

$V'(x) = 12x^2 - 92x + 120$
$\quad = 4(3x^2 - 23x + 30)$
$\quad = 4(x-5)(3x-5) = 0$
when $\cancel{x=5}$ $x = 5/3$



$V\left(\frac{5}{3}\right) = \left(15 - \frac{10}{3}\right)\left(8 - \frac{10}{3}\right)\left(\frac{5}{3}\right)$
$\quad = \left(\frac{35}{3}\right)\left(\frac{14}{3}\right)\left(\frac{5}{3}\right) = \frac{2450}{27} \approx 90.741 \text{ in}^3$

2. A tank with a rectangular base and rectangular sides is to be open at the top. It is to be constructed so that its width is 4 meters and its volume is 36 cubic meters. If building the tank costs $10 per square meter for the base and $5 per square meter for the sides, what is the cost of the least expensive tank?



$V = lwh = 4lh = 36 \therefore lh = 9 \quad l = \frac{9}{h}$

$C(l,h) = 10(4l) + 5(2lh) + 5(2\cdot 4h)$
$\quad = 40l + 90 + 40h$

$\rightarrow C(h) = \frac{360}{h} + 90 + 40h$

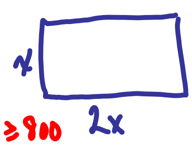
$C'(h) = -\frac{360}{h^2} + 40 = 0$ when $\frac{40}{1} = \frac{360}{h^2}$

$40h^2 = 360$
$h^2 = 9$
$h = 3$

$V = lwh = l\cdot 4\cdot 3 = 36$
$l = 3$

$C(3) = \frac{360}{3} + 90 + 40(3)$
$= 120 + 90 + 120$
$= \$330$

3. A man has 340 yard of fencing for enclosing two separate fields, one of which is to be a rectangle twice as long as it is wide and the other a square. The square field must contain at least 100 square yards and the rectangular one must contain at least 800 square yards.



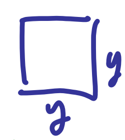
$P = 6x + 4y = 340 \qquad A = 2x^2 + y^2$

$2x^2 \geq 800$

(a) If x is the width of the rectangular field, what are the maximum and minimum possible values of x?

$20 \leq x \leq 50 \qquad x = 50 \quad x = 20 \qquad 10 \leq y \leq 55$

(b) What is the greatest number of square yards that can be enclosed in the two fields? Justify your answer.



$y^2 \geq 100$

min value of $y = 10$
$6x = 300$
$x = 50$

min value of x when
$2x^2 = 800$
$x^2 = 400 \quad x = 20$

$4y = 220$
$y = 55$

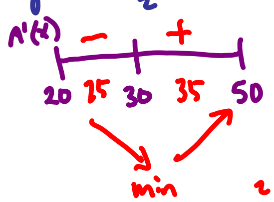
$A(x,y) = 2x^2 + y^2$

$A(x) = 2x^2 + \left(85 - \frac{3x}{2}\right)^2$

$A'(x) = 4x + \cancel{2}\left(85 - \frac{3x}{2}\right)\left(-\frac{3}{\cancel{2}}\right)$
$\quad = 4x - 3\left(85 - \frac{3x}{2}\right)$
$\quad = 4x - 255 + \frac{9x}{2}$
$\quad = \frac{17x}{2} - 255 = 0$ when

$\frac{17x}{2} = \frac{255}{1} \Rightarrow 17x = 510$
$x = 30$

$4y = 340 - 6x$
$y = 85 - \frac{3x}{2}$



$A(20) = 3825 \text{ yds}^2$
$A(30) = 3400 \text{ yds}^2$
$A(50) = 5100 \text{ yds}^2$

max yards!!

4 A particle moves on the x-axis in such a way that its position at time t is given by $x(t) = \underbrace{(2t-1)}_{1}\underbrace{(t-1)^2}_{2}$.

b) Moves left when $v(t)<0$, i.e. $\frac{2}{3}<x<1$

(a) At what times t is the particle at rest? Justify your answer.

(b) During what interval of time is the particle moving to the left? Justify your answer.

(c) At what time during the interval found in part (b) is the particle moving most rapidly (that is, the speed is a maximum)? Justify your answer.

a) At rest when $v(t)=0$, i.e. when $t=1,\ t=\frac{2}{3}$

$x'(t) = (2t-1)2(t-1)'(1) + (t-1)^2(2)$
$= (t-1)[4t-2+2t-2]$
$= (t-1)(6t-4)$
→ $= 2(t-1)(3t-2) = 0$
$t=1 \quad t=\frac{2}{3}$

c) find the max abs. value of velocity, i.e $t=\frac{5}{6}$

$x''(t) = 2(t-1)(3) + (3t-2)(2)$
$= 6t-6+6t-4$
$= 12t-10$
$= 2(6t-5) = 0$ when
$t=5/6$

5 A particle moves along the x-axis in such a way that its position at time t is given by $x = 3t^4 - 16t^3 + 24t^2$ for $-5 \le t \le 5$.

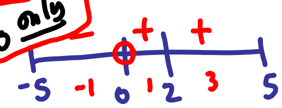
(a) Determine the velocity and acceleration of the particle at time t.

(b) At what values of t is the particle at rest?

b) At rest when $v(t)=0$, i.e. when $\boxed{t=0,\ t=2}$

(c) At what values of t does the particle change direction?

c) changes direction when $v(t)$ changes sign, i.e. $\boxed{t=0 \text{ only}}$



a) (d) What is the velocity when the acceleration is first zero?

$v(t) = x'(t) = \boxed{12t^3 - 48t^2 + 48t}$
$= 12t(t^2-4t+4)$
$= 12t(t-2)^2$ ←

$a(t) = \boxed{36t^2 - 96t + 48}$
$= 12(3t^2-8t+4)$
$= 12(3t-2)(t-2) = 0$
when $t=\frac{2}{3},\ t=2$

d) $a(t)$ is first zero at $t=\frac{2}{3}$

$v(\frac{2}{3}) = 12(\frac{2}{3})(\frac{2}{3}-2)^2$
$= 8(-\frac{4}{3})^2$
$= \frac{8\cdot 16}{9} = \boxed{\frac{128}{9}}$

6 A particle moves along the x-axis in such a way that its position at time t for $t \ge 0$ is given by $x = \frac{1}{3}t^3 - 3t^2 + 8t$.

$v(t) = t^2 - 6t + 8 = (t-4)(t-2) = 0$ when $t=4,\ t=2$

(a) Show that at time $t=0$, the particle is moving to the right. $v(0)=8$ ∴ moving right

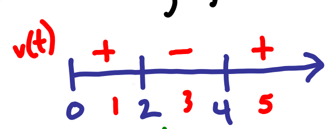
(b) Find all values of t for which the particle is moving to the left.

Moves left when $v(t)<0$, i.e $2<t<4$

(c) What is the position of the particle at time $t=3$?

$x(3) = \frac{1}{3}(27) - 3(9) + 24 = 6$

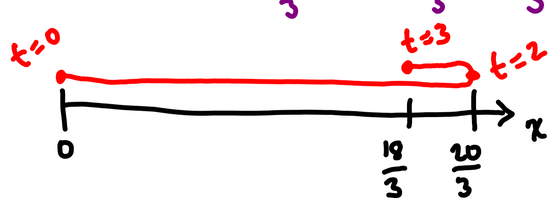
(d) When $t=3$, what is the total distance the particle has traveled?



particle started at $t=0$ and moved right until $t=2$ it then turned around and moved left until $t=3$

At $t=0 \quad x(0)=0$

$t=2 \quad x(2) = \frac{1}{3}(2)^3 - 3(2)^2 + 8(2)$
$= \frac{8}{3} - 12 + 16 = \frac{8}{3} + 4 = \frac{20}{3}$



At $t=3$, $x(3) = \frac{1}{3}(3)^3 - 3(3)^2 + 8(3)$
$= 9 - 27 + 24$
$= 6 = \frac{18}{3}$

$\frac{20}{3} + \frac{2}{3} = \frac{22}{3}$ total distance traveled

7 Consider $f(x)=\cos^2 x+2\cos x$ over one complete period beginning with $x=0$.

(a) Find all values of x in this period at which $f(x)=0$.

$f(x)=\cos x(\cos x+2)=0$ when $x=\frac{\pi}{2}, \frac{3\pi}{2}$ only

(b) Find all values of x in this period at which the function has a minimum. Justify your answer.

f(x) has a minimum when $f'(x)$ changes from negative to positive $\searrow\cup\nearrow$

(c) Over what intervals in this period is the curve concave up? i.e. when $x=\pi$

$$f'(x)=-2\cos x\sin x-2\sin x$$
$$=-2\sin x(\cos x+1)$$
$x=0$ $\quad x=\pi$

$f'(x)$: $-$ | $+$ on $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$; min at π

concave up when $f''(x)>0$

$2\sin x\cos x=\sin 2x$ $\quad \cos 2x=\cos^2 x-\sin^2 x$

$\sin^2 x+\cos^2 x=1$

$$f'(x)=-\sin 2x-2\sin x$$
$$f''(x)=-2\cos 2x-2\cos x$$
$$=-2(\cos^2 x-\sin^2 x+\cos x)$$
$$=-2(\cos^2 x-(1-\cos^2 x)+\cos x)$$
$$=-2(2\cos^2 x+\cos x-1)$$
$$=-2(2\cos x-1)(\cos x+1)=0 \text{ when}$$

$\cos x=\frac{1}{2}$ $\quad \cos x=-1$

$x=\frac{\pi}{3},\frac{5\pi}{3}$ $\quad x=\pi$

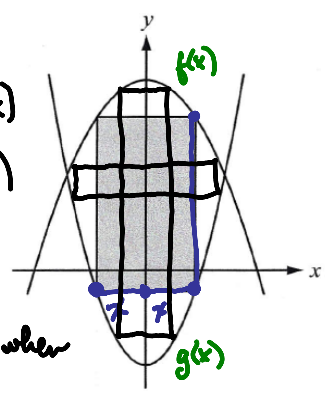
i.e. $\frac{\pi}{3}<x<\frac{5\pi}{3}$

$f''(x)$: $-$ | $+$ | $+$ | $-$ on $0, \frac{\pi}{6}, \pi/3, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 5\pi/3, \frac{11\pi}{6}, 2\pi$

$f''(x)>0$ when $\frac{\pi}{3}<x<\frac{5\pi}{3}$ so f(x) concave up there.

8 Find the area of the largest rectangle (with sides parallel to the coordinate axes) that can be inscribed in the region enclosed by the graphs of $f(x)=18-x^2$ and $g(x)=2x^2-9$.

$$A=lw$$
$$=(f(x)-g(x))2x$$
$$=(18-x^2-(2x^2-9))(2x)$$
$$=(18-x^2-2x^2+9)(2x)$$
$$=(27-3x^2)(2x)$$
$$=54x-6x^3$$



$18-x^2=2x^2-9$

$27=3x^2$

$9=x^2$

$x=\pm 3$

$0\le x\le 3$

$$A'(x)=54-18x^2=0 \text{ when}$$
$$=18(3-x^2)=0$$
$$x=\pm\sqrt{3}$$

$f(0)=0$

$f(\sqrt{3})=54\sqrt{3}-6(\sqrt{3})^3=36\sqrt{3}$

$f(3)=0$

$A'(x)$: $+$ | $-$ on $0, 1, \sqrt{3}, 2, 3$; MAX at $\sqrt{3}$

9 Given the function defined by $y=x+\sin x$ for all x such that $-\frac{\pi}{2}\le x\le\frac{3\pi}{2}$.

(a) Find the coordinates of all maximum and minimum points on the given interval. Justify your answers.

ABS MIN $\left(-\frac{\pi}{2},-\frac{\pi}{2}-1\right)$

Abs. Max and min occur at critical numbers or endpoints.

$\left(\frac{3\pi}{2},\frac{3\pi}{2}-1\right)$ ABS MAX

(b) Find the coordinates of all points of inflection on the given interval. Justify your answers. P.O.I. when $f''(x)$ changes sign, i.e. @ $x=0, x=\pi$

$f'(x)=1+\cos x=0$ when

$\cos x=-1$ $\quad$ Note: $f'(x)\ge 0$ on the interval

$x=\pi$

$f''(x)=-\sin x=0$ @ $x=0, x=\pi$

$f''(x)$: $+$ | $-$ | $+$ on $-\frac{\pi}{2}, -\frac{\pi}{6}, 0, \frac{\pi}{2}, \pi, \frac{7\pi}{6}, \frac{3\pi}{2}$

$f\left(-\frac{\pi}{2}\right)=-\frac{\pi}{2}+\sin\left(-\frac{\pi}{2}\right)=-\frac{\pi}{2}-1 \leftarrow$ MIN

$f(\pi)=\pi+\sin(\pi)=\pi$

$f\left(\frac{3\pi}{2}\right)=\frac{3\pi}{2}+\sin\left(\frac{3\pi}{2}\right)=\frac{3\pi}{2}-1\leftarrow$ MAX

$y = x + \sin x$

(c) On the axes provided, sketch the graph of the function.

ABS MIN: $\left(-\frac{\pi}{2}, -\frac{\pi}{2}-1\right)$

STATIONARY PT: (π, π)

ABS MAX: $\left(\frac{3\pi}{2}, \frac{3\pi}{2}-1\right)$

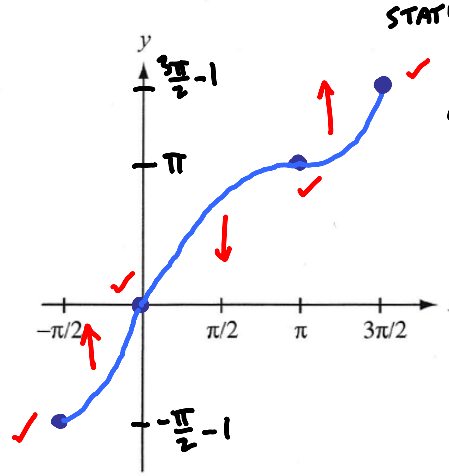
INCREASING FOR ALL X

CONCAVE UP: $-\frac{\pi}{2} < x < 0$

$\pi < x < \frac{3\pi}{2}$

CONCAVE DOWN: $0 < x < \pi$

(0,0) is on the curve



10

Consider the function f given by $f(x) = x^{\frac{4}{3}} + 4x^{\frac{1}{3}}$ on the interval $-8 \le x \le 8$.

(a) Find the coordinates of all points at which the tangent to the curve is a horizontal line.

$f'(x) = 0$ when $x = -1$ $y = -3$

(b) Find the coordinates of all points at which the tangent to the curve is a vertical line.

$f'(x)$ DNE when $x = 0$ $y = 0$

(c) Find the coordinates of all points at which the absolute maximum and absolute minimum occur.

Abs min and max occur at either critical values or end points,

abs min @ $(-1, -3)$

abs max @ $(8, 24)$

(d) For what values of x is this function concave down?

$f(-8) = 8$

$f(-1) = -3$

$f(0) = 0$

$f(8) = 24$

$$f'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3} = 0$$

$$= \frac{4}{3}x^{-2/3}\left(x^{3/3} + 1\right) = 0$$

$$= \frac{4}{3}x^{-2/3}(x+1) = 0 \text{ when } x = -1$$

but DNE when $x = 0$

Note: $x = -1$ and $x = 0$ are both critical values of $f(x)$

$$f''(x) = \frac{4}{9}x^{-2/3} - \frac{8}{9}x^{-5/3} = 0$$

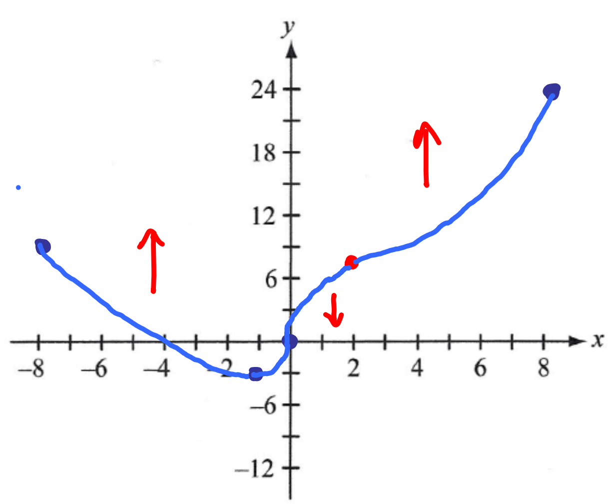
$$= \frac{4}{9}x^{-5/3}(x - 2) = 0 \text{ when } x = 2$$

but DNE when $x = 0$

$f''(x)$: $+$ on $(-8, 0)$ [test -1], $-$ on $(0, 2)$ [test 1], $+$ on $(2, 8)$ [test 3]

$f(x)$ concave down for all $0 < x < 2$.

(e) On the axes provided, sketch the graph of the function on this interval.



$f(-8) = 8$

$f(-1) = -3$

$f(0) = 0$

$f(8) = 24$

$f'(-1) = 0$

$f''(-1) > 0$

$\therefore$ min

vertical tangent @ (0,0)

11 A function f is [redacted] on the closed interval $[-3, 3]$ such that $f(-3)=4$ and $f(3)=1$. The functions f' and f'' have the properties given in the table below.

$f(-3)=4$
$f(-1)=?$ ← abs max
$f(1)=?$
$f(3)=1$ ← abs min

x	$-3<x<-1$	$x=-1$	$-1<x<1$	$x=1$	$1<x<3$
$f'(x)$	Positive	Fails to exist	Negative	0	Negative
$f''(x)$	Positive	Fails to exist	Positive	0	Negative

f(x) only decreases when $-1<x<3$. Since $f(3)<f(-3)$, abs. min @ $x=3$.

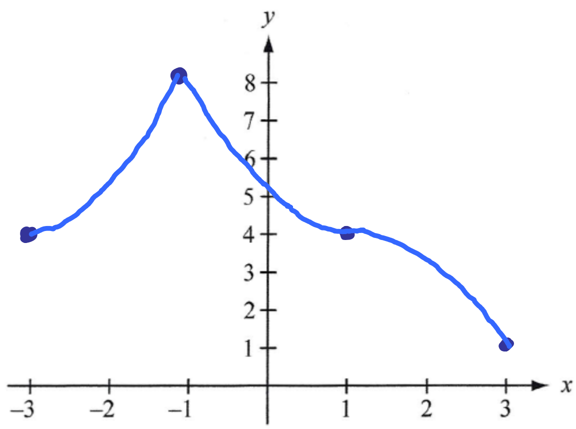
(a) What are the x-coordinates of all absolute maximum and absolute minimum points of f on the interval $[-3, 3]$? Justify your answer.

Abs. max and min occur at critical numbers and end points.
f(x) increases until $x=-1$ and then decreases from there, so abs max at $x=-1$

(b) What are the x-coordinates of all points of inflection of f on the interval $[-3, 3]$? Justify your answer.

P.O.I. occur when $f''(x)$ changes sign, i.e. @ $x=1$ only

(c) On the axes provided, sketch a graph that satisfies the given properties of f.



12 Consider the function f defined by $f(x)=(x^2-1)^3$ for all real numbers x.

f(x) has zeros at $x=\pm 1$

(a) For what values of x is the function increasing?

f(x) increases when $f'(x)>0$, i.e. for all $x>0$

(b) Find the x- and y-coordinates of the relative maximum and minimum points. Justify your answer. Rel. max when $f'(x)$ changes from + to −, no such points.
Rel. min when $f'(x)$ changes from − to +, i.e. @ $x=0$, $y=-1$

(c) For what values of x is the graph of f concave upward?

$f'(x)=3(x^2-1)^2(2x)$
$=6x[(x+1)(x-1)]^2=0$ when
$x=0 \quad x=-1 \quad x=1$

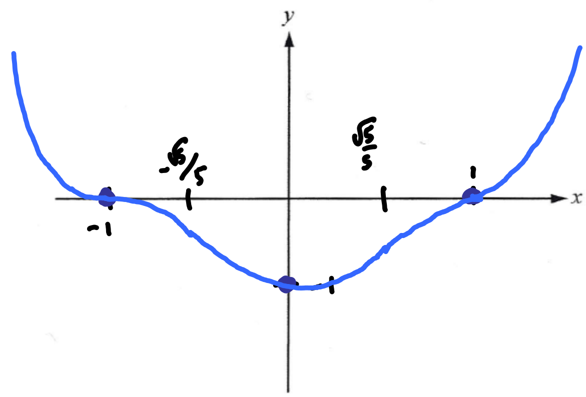
$f'(x)$: − − + + on $-2 \; -1 \; -\frac{1}{2} \; 0 \; \frac{1}{2} \; 1 \; 2$, min at 0

$f'(x)=6x(x^2-1)^2$
$f''(x)=6x\cdot 2(x^2-1)^1(2x)+6(x^2-1)^2$
$=6(x^2-1)[4x^2+x^2-1]$
$=6(x+1)(x-1)(5x^2-1)$
$x=-1 \quad x=1 \quad x=\pm\frac{\sqrt{5}}{5}$

$f''(x)$: + − + − + on $-2 \; -1 \; -\frac{3}{4} \; -\frac{\sqrt{5}}{5} \; 0 \; \frac{\sqrt{5}}{5} \; \frac{3}{4} \; 1 \; 2$

f(x) is concave down on $(-1, -\frac{\sqrt{5}}{5}) \cup (\frac{\sqrt{5}}{5}, 1)$

(d) Using the information found in parts (a), (b), and (c), sketch the graph of f on the axes provided.



13 Let f be the function defined by $f(x) = 12x^{\frac{2}{3}} - 4x$.

$f'(x) = \frac{2}{3} 12x^{-1/3} - 4$

$= 8x^{-1/3} - 4$

$= 4(2x^{-1/3} - 1) = 0$ when $2x^{-1/3} = 1$

$\frac{2}{x^{1/3}} = \frac{1}{1} \Rightarrow x^{1/3} = 2 \Rightarrow x = 8$

(a) Find the intervals on which f is increasing.

$f(x)$ increases when $f'(x) > 0$, i.e. for $0 < x < 8$

(b) Find the x- and y-coordinates of all relative maximum points.

rel max when $f'(x)$ changes + to −, i.e. $x = 8$, $y = 16$

(c) Find the x- and y-coordinates of all relative minimum points.

rel min when $f'(x)$ changes − to +, i.e. $x = 0$, $y = 0$

$f'(x)$ − + −

-1 0 1 8 27

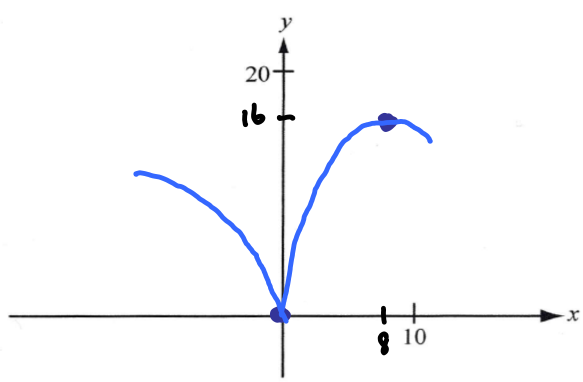
(d) Find the intervals on which f is concave downward.

concave down when $f''(x) < 0$

$f'(x) = 8x^{-1/3} - 4$

$f''(x) = -\frac{8}{3}x^{-4/3}$ $\therefore$ $f(x)$ is concave down for all x, $x \neq 0$

(e) Using the information found in parts (a), (b), (c), and (d), sketch the graph of f on the axes provided.



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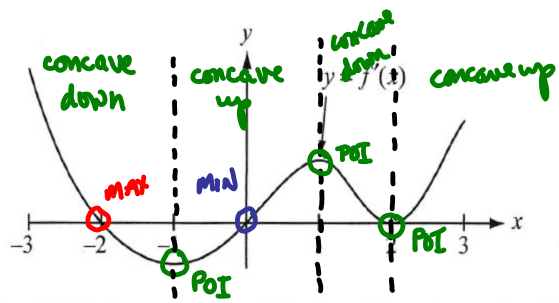

Note: This is the graph of the <u>derivative</u> of f, <u>not</u> the graph of f.

The figure above shows the graph of f', the derivative of a function f. The domain of the function f is the set of all x such that $-3 \le x \le 3$.

(a) For what values of x, $-3 < x < 3$, does f have a relative maximum? A relative minimum? Justify your answer. Rel. max when $f'(x)$ changes + to −, i.e. $x = -2$
Rel. min when $f'(x)$ changes − to +

(b) For what values of x is the graph of f concave up? Justify your answer.
$f(x)$ is concave up when $f''(x) > 0$, i.e. when $f'(x)$ increases, i.e. $(-1,1) \cup (2,3)$

(c) Use the information found in parts (a) and (b) and the fact that $f(-3) = 0$ to sketch a possible graph of f on the axes provided below.