Answers for Set 4: Applications of Differential Calculus

1. D	**21.** D	**41.** A	**61.** C	**81.** D
2. A	**22.** A	**42.** D	**62.** D	**82.** C
3. E	**23.** C	**43.** C	**63.** A	**83.** D
4. B	**24.** B	**44.** D	**64.** E	**84.** E
5. A	**25.** E	**45.** C	**65.** A	**85.** A
6. D	**26.** C	**46.** E	**66.** B	**86.** B
7. E	**27.** A	**47.** B	**67.** C	**87.** A
8. C	**28.** B	**48.** A	**68.** E	**88.** C
9. D	**29.** E	**49.** C	**69.** C	**89.** D
10. C	**30.** D	**50.** E	**70.** D	**90.** C
11. E	**31.** B	**51.** A	**71.** B	**91.** E
12. B	**32.** E	**52.** B	**72.** B	**92.** E
13. D	**33.** C	**53.** D	**73.** B	
14. A	**34.** E	**54.** D	**74.** D	
15. C	**35.** B	**55.** E	**75.** C	
16. B	**36.** D	**56.** B	**76.** A	
17. D	**37.** A	**57.** D	**77.** A	
18. D	**38.** E	**58.** D	**78.** E	
19. D	**39.** B	**59.** C	**79.** D	
20. E	**40.** D	**60.** B	**80.** E	

1. D. Substituting $y = 2$ yields $x = 1$. We find y' implicitly.

$$3y^2y' - (2xyy' + y^2) = 0; \quad (3y^2 - 2xy)y' - y^2 = 0.$$

Replace x by 1 and y by 2; solve for y'.

2. A. $2yy' - (xy' + y) - 3 = 0$. Replace x by 0 and y by -1; solve for y'.

3. E. Find the slope of the curve at $x = \frac{\pi}{2}$: $y' = x\cos x + \sin x$. At $x = \frac{\pi}{2}$, $y' = \frac{\pi}{2} \cdot 0 + 1$. The equation is $y - \frac{\pi}{2} = 1\left(x - \frac{\pi}{2}\right)$.

4. B. Since $y' = e^{-x}(1 - x)$ and $e^{-x} > 0$ for all x, $y' = 0$ when $x = 1$.

5. A. Since the slope of the tangent to the curve is $y' = \frac{1}{\sqrt{2x+1}}$, the slope of the normal is $-\sqrt{2x+1}$. So $-\sqrt{2x+1} = -3$ and $2x + 1 = 9$.

6. D. The slope $y' = 5x^4 + 3x^2 - 2$. Let $g = y'$. Since $g'(x) = 20x^3 + 6x = 2x(10x^2 + 3)$, $g'(x) = 0$ only if $x = 0$. Since $g''(x) = 60x^2 + 6$, g'' is always positive, assuring that $x = 0$ yields the minimum slope. Find y' when $x = 0$.

7. E. The slope $y' = \frac{2x}{4}$; at the given point $y' = -\frac{4}{4} = -1$ and $y = 1$. The equation is therefore

$$y - 1 = -1(x + 2) \qquad \text{or} \qquad x + y + 1 = 0.$$

8. C. Since $2x - 2yy' = 0$, $y' = \frac{x}{y}$. At (4, 2), $y' = 2$. The equation of the tangent at (4, 2) is $y - 2 = 2(x - 4)$.

9. D. Since $y' = \frac{y}{2y - x}$, the tangent is vertical for $x = 2y$. Substitute in the given equation.

10. C. $a \approx \frac{\Delta v}{\Delta t} = \frac{v(8) - v(4)}{8 - 4} = \frac{10 - 16}{4}$ ft/sec^2.

11. E. Since $f' < 0$ on $5 \leqslant x < 7$, the function decreases as it approaches the right endpoint.

12. B. For $x < 5$, $f' > 0$, so f is increasing; for $x > 5$, f is decreasing.

13. D. f being concave downward implies that $f'' < 0$, which implies that f' is decreasing.

14. A. Since $V = \frac{4}{3}\pi r^3$, therefore $V'(r) = 4\pi r^2$. The linearization equation gives the approximate increase in volume as $V'(3)(0.1)$ or 11.310 in.^{3}

15. C. Differentiating implicitly yields $4x - 3y^2y' = 0$. So $y' = \frac{4x}{3y^2}$. The linear approximation for the true value of y when x changes from 3 to 3.04 is

$$y_{\text{at } x=3} + y'_{\text{at point } (3,2)} \cdot (3.04 - 3).$$

Since it is given that, when $x = 3$, $y = 2$, the approximate value of y is

$$2 + \frac{4x}{3y^2}_{\text{at } (3,2)} \cdot (0.04)$$

or

$$2 + \frac{12}{12} \cdot (0.04) = 2.04.$$

16. B. We want to approximate the change in area of the square when a side of length e increases by $0.01e$. The answer is

$$A'(e)(0.01e) \text{ or } 2e(0.01e).$$

17. D. Linear approximation yields $V'(10)(\pm 0.1)$. Since $V = e^3$, we get $3e^2$ (at $e = 10$) times ± 0.1, which equals $300(\pm 0.1) = 30$ in.^{3}

18. D. Speed is the magnitude of velocity; at $t = 3$, speed = 10 ft/sec.

19. D. Speed increases from 0 at $t = 2$ to 10 at $t = 3$; it is constant or decreasing elsewhere.

20. E. Acceleration is positive when the *velocity* increases.

21. D. Acceleration is undefined when velocity is not differentiable. Here that occurs at $t = 1, 2, 3$.

22. A. Acceleration is the derivative of velocity. Since the velocity is linear, its derivative is its slope.

23. C. Positive velocity implies motion to the right ($t < 2$); negative velocity ($t > 2$) implies motion to the left.

24. B. The average rate of change of velocity is $\frac{v(3) - v(0)}{3 - 0} = \frac{-10 - 5}{-3}$ ft/sec^2.

25. E. $f'(x) = 4x^3 - 8x = 4x(x^2 - 2)$. $f' = 0$ if $x = 0$ or $\pm\sqrt{2}$.
$f''(x) = 12x^2 - 8$; f'' is positive if $x = \pm\sqrt{2}$, negative if $x = 0$.

26. C. Since $f''(x) = 4(3x^2 - 2)$, it equals 0 if $x = \pm\sqrt{\frac{2}{3}}$. Since f'' changes sign as increases through each of these, both are inflection points.

27. A. The domain of y is $\{x \mid x \leqq 2\}$. Note that y is negative for each x in the domain except 2, where $y = 0$.

28. B. $f'(x)$ changes sign only as x passes through zero.

29. E. The derivative equals $x \cos \frac{1}{x}\left(-\frac{1}{x^2}\right) + \sin \frac{1}{x}$.

30. D. Since $f'(x) = 4 \cos x + 3 \sin x$, the critical values of x are those for which tan $x = -\frac{4}{3}$. For these values,

when $\frac{\pi}{2} < x < \pi$, then $\sin x = \frac{4}{5}$ and $\cos x = -\frac{3}{5}$;

but

when $\frac{3\pi}{2} < x < 2\pi$, then $\sin x = -\frac{4}{5}$ and $\cos x = \frac{3}{5}$.

If these are used to determine the sign of $f''(x)$, we see that the second-quadrant x yield a negative second derivative, but the fourth-quadrant x give a positive second derivative. It follows that the function is a maximum for the former set, for which f has the value 5. Note that $f(0) = f(2\pi) = -3$.

31. B. We find the points of intersection. From the first curve we get $y = \frac{2}{x}$, which yields, in the second curve,

$$x^2 - \frac{4}{x^2} = 3 \quad \text{or} \quad x^4 - 3x^2 - 4 = 0.$$

$(x^2 - 4)(x^2 + 1) = 0$ if $x = \pm 2$. So the points of intersection are (2, 1) and (−2, −1). Since $m_1 = -\frac{2}{x^2}$ and $m_2 = \frac{x}{y}$, at each point $m_1 = -\frac{1}{2}$ and $m_2 = 2$

32. E. The slope of $y = x^3$ is $3x^2$. It is equal to 3 when $x = \pm 1$. At $x = 1$, the equation of the tangent is

$$y - 1 = 3(x - 1) \qquad \text{or} \qquad y = 3x + 2.$$

At $x = -1$, the equation is

$$y + 1 = 3(x + 1) \qquad \text{or} \qquad y = 3x + 2.$$

33. C. Let the tangent to the parabola from (3, 5) meet the curve at (x_1, y_1). Its equation is $y - 5 = 2x_1(x - 3)$. Since the point (x_1, y_1) is on both the tangent and the parabola, we solve simultaneously:

$$y_1 - 5 = 2x_1(x_1 - 3) \qquad \text{and} \qquad y_1 = x_1^2$$

The points of tangency are (5, 25) and (1, 1). The slopes, which equal $2x_1$, are 10 and 2.

34. E. $a \approx \dfrac{\Delta v}{\Delta t} = \dfrac{v(1.5) - v(1.0)}{0.5} = \dfrac{13.2 - 12.2}{0.5}$ ft/sec².

35. B. The distance is increasing when v is positive. Since $v = \dfrac{ds}{dt} = 3(t-2)^2$, $v > 0$ for all $t \neq 2$.

36. D. The speed $= |v|$. From Question 35, $|v| = v$. The least value of v is 0.

37. A. The acceleration $a = \dfrac{dv}{dt}$. From Question 35, $a = 6(t - 2)$.

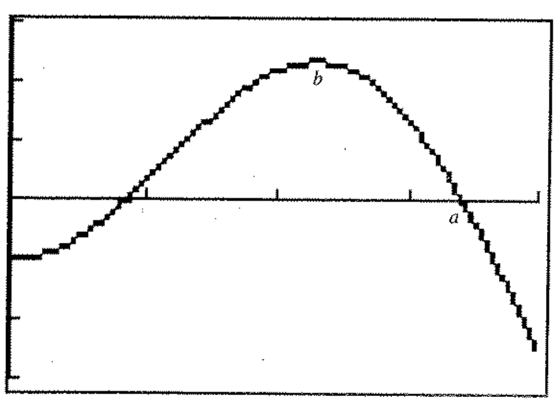
38. E. The speed is decreasing when v and a have opposite signs. The answer is $t < 2$, since for all such t the velocity is positive while the acceleration is negative. For $t > 2$, both v and a are positive.

39. B. The particle is at rest when $v = 0$; $v = 2t(2t^2 - 9t + 12) = 0$ only if $t = 0$. Note that the discriminant of the quadratic factor $(b^2 - 4ac)$ is negative.

40. D. Since $a = 12(t - 1)(t - 2)$, we check the signs of a in the intervals $t < 1$, $1 < t < 2$, and $t > 2$. We choose those where $a > 0$.

41. A. From Questions 39 and 40 we see that $v > 0$ if $t > 0$ and that $a > 0$ if $t < 1$ or $t > 2$. So both v and a are positive if $0 < t < 1$ or $t > 2$. There are no values of t for which both v and a are negative.

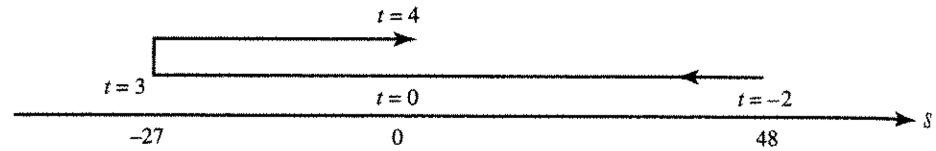
42. D. The graph of $f'(x) = x \sin x - \cos x$ is drawn here in the window $[0,4] \times [-3,3]$:



A local maximum exists where f' changes from positive to negative; use `TRACE` to approximate a.

43. C. f'' changes sign when f' changes from increasing to decreasing (or vice versa). Again, use TRACE to find the abscissa at b.

44. D. See the figure, which shows the motion of the particle during the time interval $-2 \leqslant t \leqslant 4$. The particle is at rest when $t = 0$ or 3, but reverses direction only at 3. The endpoints need to be checked here, of course. Indeed, the maximum displacement occurs at one of those, namely, when $t = -2$.



45. C. Since $v = 5t^3(t + 4)$, $v = 0$ when $t = -4$ or 0. Note that v does change sign at each of these times.

46. E. Eliminating t yields the equation $y = -\frac{1}{4}x^2 + 2x$.

47. B. $|\mathbf{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{2^2 + (4-2t)^2}$.

48. A. Since $|\mathbf{v}| = 2\sqrt{t^2 - 4t + 5}$, $\frac{d|\mathbf{v}|}{dt} = \frac{2t-4}{\sqrt{t^2-4t+5}} = 0$ if $t = 2$. We note that, as t increases through 2, the signs of $|\mathbf{v}|'$ are –, 0, +, assuring a minimum of $|\mathbf{v}|$ at $t = 2$. Evaluate $|\mathbf{v}|$ at $t = 2$.

49. C. The direction of $\mathbf{a}$ is $\tan^{-1} \dfrac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}$. Since $\frac{d^2x}{dt^2} = 0$ and $\frac{d^2y}{dt^2} = -2$, the acceleration is always directed downward. Its magnitude, $\sqrt{0^2 + (-2)^2}$, is 2 for all t.

50. E. Since $x = 3 \cos \frac{\pi}{3} t$ and $y = 2 \sin \frac{\pi}{3} t$, we note that $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$.

51. A. Note that $\mathbf{v} = -\pi \sin \frac{\pi}{3} t\mathbf{i} + \frac{2\pi}{3} \cos \frac{\pi}{3} t\mathbf{j}$. At $t = 3$,

$$|\mathbf{v}| = \sqrt{(-\pi \cdot 0)^2 + \left(\frac{2\pi}{3} \cdot -1\right)^2}.$$

52. B. $\mathbf{a} = -\frac{\pi^2}{3} \cos \frac{\pi}{3} t\mathbf{i} - \frac{2\pi^2}{9} \sin \frac{\pi}{3} t\mathbf{j}$. At $t = 3$,

$$|\mathbf{a}| = \sqrt{\left(\frac{-\pi^2}{3} \cdot -1\right)^2 + \left(\frac{-2\pi^2}{9} \cdot 0\right)^2}.$$

53. D. The slope of the curve is the slope of **v**, namely, $\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$. At $t = \frac{1}{2}$, the slope is equal to

$$\frac{\frac{2\pi}{3}\cdot\cos\frac{\pi}{6}}{-\pi\cdot\sin\frac{\pi}{6}} = -\frac{2}{3}\cot\frac{\pi}{6}.$$

54. D. Using the notations v_x, v_y, a_x, and a_y, we are given that $|\mathbf{v}| = \sqrt{v_x^2 + v_y^2} = k$, where k is a constant. Then

$$\frac{d|\mathbf{v}|}{dt} = \frac{v_x a_x + v_y a_y}{|\mathbf{v}|} = 0 \quad \text{or} \quad \frac{v_x}{v_y} = -\frac{a_y}{a_x}.$$

55. E. $\dfrac{dy}{dt} = \left(2 - \dfrac{1}{x}\right)\dfrac{dx}{dt} = \left(2 - \dfrac{1}{x}\right)(-2)$.

56. B. A local minimum exists where f changes from decreasing ($f' < 0$) to increasing ($f' > 0$). Note that f has local maxima at both endpoints, $x = 0$ and $x = 5$.

57. D. See Answer 43.

58. D. f' changes from increasing ($f'' > 0$) to decreasing ($f'' < 0$). Note that f is differentiable at a (because $f'(a)$ exists) and therefore continuous at a.

59. C. Since $V = \frac{4}{3}\pi r^3$, $\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$. Since $\dfrac{dV}{dt} = 4$, $\dfrac{dr}{dt} = \dfrac{1}{\pi r^2}$. When $V = \dfrac{32\pi}{3}$, $r = 2$ and $\dfrac{dr}{dt} = \dfrac{1}{4\pi}$.

$$S = 4\pi r^2; \qquad \frac{dS}{dt} = 8\pi r\frac{dr}{dt};$$

when $r = 2$, $\dfrac{dS}{dt} = 8\pi(2)\left(\dfrac{1}{4\pi}\right) = 4$.

60. B. See Figure N4–23 on page 124. Replace the printed measurements of the radius and height by 10 and 20, respectively. We are given here that $r = \dfrac{h}{2}$ and that $\dfrac{dh}{dt} = -\dfrac{1}{2}$. Since $V = \frac{1}{3}\pi r^2 h$, we have $V = \dfrac{\pi}{3}\dfrac{h^3}{4}$, so

$$\frac{dV}{dt} = \frac{\pi h^2}{4}\frac{dh}{dt} = \frac{-\pi h^2}{8}.$$

Replace h by 8.

61. C. We know that $\dfrac{dh}{dt} = \dfrac{dr}{dt} = 2$. Since $S = 2\pi rh$,

$$\frac{dS}{dt} = 2\pi\left(r\frac{dh}{dt} + h\frac{dr}{dt}\right).$$

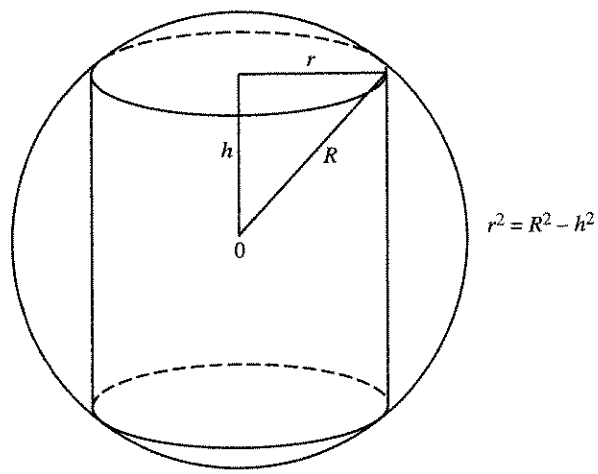
62. D. $y' = \dfrac{e^x(x-1)}{x^2}$ $(x \neq 0)$. Since $y' = 0$ if $x = 1$ and changes from negative to positive as x increases through 1, $x = 1$ yields a minimum. Evaluate y at $x = 1$.

63. A. The domain of y is $-\infty < x < \infty$. The graph of y, which is nonnegative, is symmetric to the y-axis. The inscribed rectangle has area $A = 2xe^{-x^2}$. Thus $A' = \frac{2(1-2x^2)}{e^{x^2}}$, which is 0 when the positive value of x is $\frac{\sqrt{2}}{2}$. This value of x yields maximum area. Evaluate A.

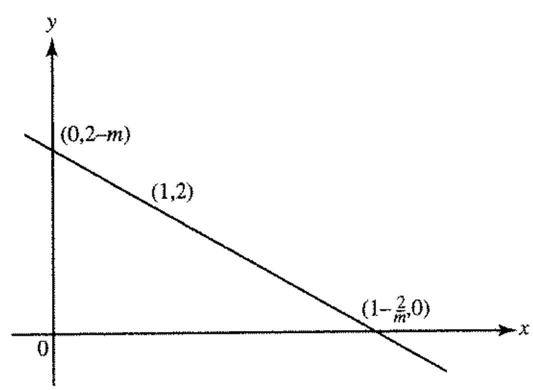
64. E. The equation of the tangent is $y = -2x + 5$. Its intercepts are $\frac{5}{2}$ and 5.

65. A. See the figure, where V, the volume of the cylinder, equals $2\pi r^2 h$. Then $V = 2\pi(R^2h - h^3)$ is a maximum for $h^2 = \frac{R^2}{3}$, and the ratio of the volumes of sphere to cylinder is

$$\frac{4}{3}\pi R^3 : 2\pi \cdot \frac{2}{3}R^2 \cdot \frac{R}{\sqrt{3}} \quad \text{or} \quad \sqrt{3}:1.$$



66. B. See the figure. If we let m be the slope of the line, then its equation is $y - 2$ $m(x - 1)$ with intercepts as indicated in the figure.



The area A of the triangle is given by

$$A = \frac{1}{2}(2-m)\left(1-\frac{2}{m}\right) = \frac{1}{2}\left(4-\frac{4}{m}-m\right).$$

Then $\frac{dA}{dm} = \frac{1}{2}\left(\frac{4}{m^2}-1\right)$ and equals 0 when $m = \pm 2$; m must be negative.

67. C. Let $q = (x-6)^2 + y^2$ be the quantity to be minimized. Then

$$q = (x-6)^2 + (x^2 - 4);$$

$q' = 0$ when $x = 3$. Note that it suffices to minimize the square of the distance.

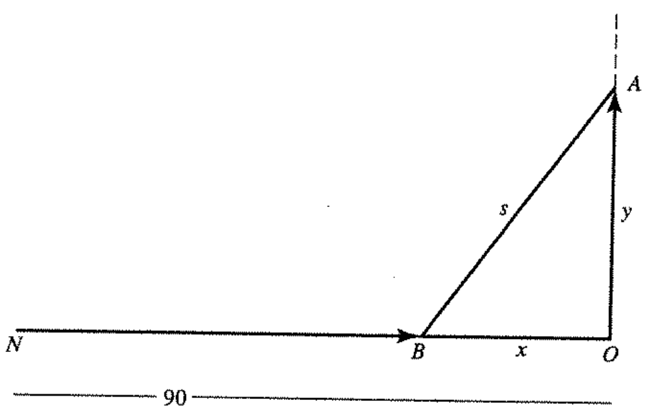
68. E. Minimize, if possible, xy, where $x^2 + y^2 = 200$ $(x, y > 0)$. The derivative of the product is $\frac{2(100 - x^2)}{\sqrt{200 - x^2}}$, which equals 0 for $x = 10$. But the signs of the derivative as x increases through 10 show that 10 yields a maximum product. No minimum exists.

69. C. Minimize $q = (x-2)^2 + \frac{18}{x}$. Since

$$q' = 2(x-2) - \frac{18}{x^2} = \frac{2(x^3 - 2x^2 - 9)}{x^2},$$

$q' = 0$ if $x = 3$. The signs of q' about $x = 3$ assure a minimum.

70. D. See the figure. At noon, car A is at O, car B at N; the cars are shown t hours after noon. We know that $\frac{dx}{dt} = -60$ and that $\frac{dy}{dt} = 40$. Using $s^2 = x^2 + y^2$, we get



$$\frac{ds}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{s} = \frac{-60x + 40y}{s}.$$

At 1 P.M., $x = 30$, $y = 40$, and $s = 50$.

71. B. $\frac{ds}{dt}$ (from Question 70) is zero when $y = \frac{3}{2}x$. Note that $x = 90 - 60t$ and $y = 40t$.

72. B. Maximum acceleration occurs when the derivative (slope) of velocity is greatest.

73. B. The object changes direction only when velocity changes sign. Velocity changes sign from negative to positive at $t = 5$.

74. D. From the graph, $f'(2) = 3$, and we are told the line passes through (2,10). We therefore have $f(x) \simeq 10 + 3(x-2) = 3x + 4$.

75. C. At $x = 1$ and 3, $f'(x) = 0$; therefore f has horizontal tangents.
For $x < 1, f' > 0$; therefore f is increasing.
For $x > 1, f' < 0$, so f is decreasing.
For $x < 2, f'$ is decreasing, so $f'' < 0$ and f is concave downward.
For $x > 2, f'$ is increasing, so $f'' > 0$ and f is concave upward.

76. A. The best approximation for $\sqrt[3]{27+h}$ when h is small is the local linear (or tangent line) approximation. If we let $f(h) = \sqrt[3]{27+h}$, then $f'(h) = \frac{1}{3(27+h)^{2/3}}$ and $f'(0) = \frac{1}{3 \cdot 9}$. The approximation for $f(h)$ is $f(0) + f'(0) \cdot h$, which equals $3 + \frac{1}{27}h$.

77. A. Since $f'(x) = e^{-x}(1-x), f'(0) > 0$.

78. E. Cite counterexamples to show that no one of (A) through (D) is necessarily true.

79. D. Since $V = 10\ell w$, $V' = 10\left(\ell\frac{dw}{dt} + w\frac{d\ell}{dt}\right) = 10(8 \cdot -4 + 6 \cdot 2)$.

80. E. We differentiate implicitly: $3x^2 + x^2y' + 2xy + 4y' = 0$. Then $y' = -\frac{3x^2+2xy}{x^2+4}$.
At (3,–2), $y' = -\frac{27-12}{9+4} = -\frac{15}{13}$.

81. D. Since $ab > 0$, a and b have the same sign; therefore $f''(x) = 12ax^2 + 2b$ never equals 0. The curve has one horizontal tangent at $x = 0$.

82. C. Note that $\frac{dy}{dx} = 0$ at Q, R, and T. At Q, $\frac{d^2y}{dx^2} > 0$; at T, $\frac{d^2y}{dx^2} < 0$.

83. D. Only at S does the graph both rise and change concavity.

84. E. Only at T is the tangent horizontal and the curve concave down.

85. A. Note that, since $y = \frac{2x^2+4}{(2-x)(1+4x)}$, both $x = 2$ and $x = -\frac{1}{4}$ are vertical asymptotes. Also, $y = -\frac{1}{2}$ is a horizontal asymptote. (See page 31.)

86. B. Since $\frac{dy}{dx} = -\frac{3t^2}{2}$, therefore, at $t = 1$, $\frac{dy}{dx} = -\frac{3}{2}$. Also, $x = 3$ and $y = 2$.

87. A. Let $f(x) = x^{1/3}$, and find the local linearization at (64, 4). Since $f'(x) = \frac{1}{3}x^{-2/3}$, $f'(64) = \frac{1}{48}$. If we move one unit to the left of 64, the tangent line will drop approximately $\frac{1}{48}$ unit.

88. C. Since $f'(6) = 4$, the equation of the tangent at (6, 30) is $y - 30 = 4(x - 6)$. Therefore $f(x) \simeq 4x + 6$ and $f(6.02) \simeq 30.08$.

In Questions 89–91, use $f(x) = f(a) + f'(a)(x - a)$ as the best linear (or tangent-line) approximation or local linearization for $f(x)$ near a.

89. D. $\tan x \simeq \tan\left(\frac{\pi}{4}\right) + \sec^2\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) = 1 + 2\left(x - \frac{\pi}{4}\right)$

90. C. $\sqrt{x^2 + 16} \simeq \sqrt{9 + 16} + \frac{(-3)}{\sqrt{(-3)^2 + 16}}(x + 3) = 5 - \frac{3}{5}(x + 3)$

91. E. $e^{kh} \simeq e^{k \cdot 0} + ke^{k \cdot 0}(h - 0) = 1 + kh$

92. E. Since the curve has a positive y-intercept, $e > 0$. Note that $f'(x) = 2cx + d$ and $f''(x) = 2c$. Since the curve is concave down, $f''(x) < 0$, implying that $c < 0$. Since the curve is decreasing at $x = 0$, $f'(0)$ must be negative, implying, since $f'(0) = d$, that $d < 0$. Therefore $c < 0$, $d < 0$, and $e > 0$.