1971 AB5

Let R be the region in the first quadrant bounded by the x-axis and the curve $y = 2x - x^2$. $x(2-x)$

(a) Find the volume produced when R is revolved about the x-axis.

(b) Find the volume produced when R is revolved about the y-axis.

DISK METHOD

$A = \pi r^2$

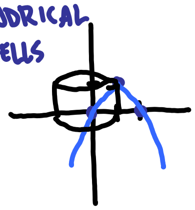
$V = \pi r^2 h$

$$V = \pi \int_0^2 (2x - x^2)^2 \, dx = \pi \int_0^2 4x^2 - 4x^3 + x^4 \, dx$$

$$= \pi \left[\frac{4x^3}{3} - x^4 + \frac{x^5}{5} \right]_0^2 = \pi \left[\left(\frac{32}{3} - 16 + \frac{32}{5} \right) - 0 \right]$$

$$= \pi \left[\frac{160}{15} - \frac{240}{15} + \frac{96}{15} \right] = \frac{16\pi}{15}$$

CYLINDRICAL SHELLS



$A = 2\pi r h$

$V = 2\pi r h \, dx$

$$V = 2\pi \int_0^2 x(2x - x^2) \, dx$$

$$= 2\pi \int_0^2 2x^2 - x^3 \, dx$$

$$= 2\pi \left[\frac{2x^3}{3} - \frac{x^4}{4} \right]_0^2$$

$$= 2\pi \left[\left(\frac{16}{3} - \frac{16}{4} \right) - 0 \right] = 2\pi \left(\frac{16}{12} \right) = \boxed{\frac{8\pi}{3}}$$

1969 AB5

Let R denote the region enclosed between the graph of $y = x^2$ and the graph of $y = 2x$.

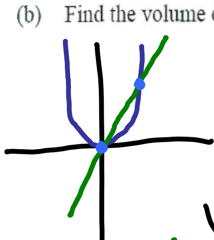
$x^2 = 2x$

$x^2 - 2x = 0$

$x(x-2) = 0$

(a) Find the area of region R.

(b) Find the volume of the solid obtained by revolving the region R about the y-axis.

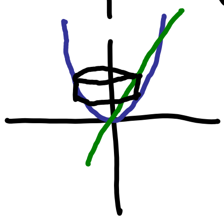


$$A = \int_0^2 2x - x^2 \, dx = \left[x^2 - \frac{x^3}{3} \right]_0^2 = \left[\left(4 - \frac{8}{3} \right) - 0 \right] = \boxed{\frac{4}{3}}$$

$$V = 2\pi \int_0^2 \underbrace{x}_{r} \underbrace{(2x - x^2)}_{h} \, dx = 2\pi \int_0^2 2x^2 - x^3 \, dx$$

$$= 2\pi \left[\frac{2x^3}{3} - \frac{x^4}{4} \right]_0^2 = 2\pi \left[\left(\frac{16}{3} - \frac{16}{4} \right) - 0 \right]$$

$$= \boxed{\frac{8\pi}{3}}$$



1975 AB6/BC2

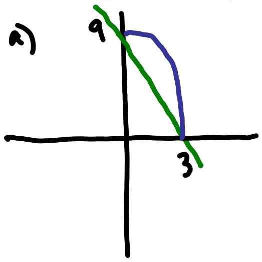
Let R be the region in the first quadrant bounded by the graphs of $\frac{x^2}{9}+\frac{y^2}{81}=1$ and $3x+y=9$.

$y=-3x+9$

$y=\sqrt{81-9x^2}$

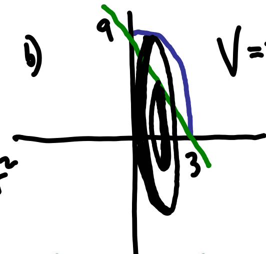
(a) Set up but do not evaluate an integral representing the area of R. Express the integrand as a function of a single variable.

a)



$$\int_0^3 \sqrt{81-9x^2}-(-3x+9)\,dx$$

b)



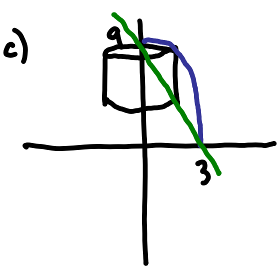
$$V=\pi\int_0^3 \left(\sqrt{81-9x^2}\right)^2-(-3x+9)^2\,dx$$

$\pi R^2-\pi r^2$

(b) Set up but do not evaluate an integral representing the volume of the solid generated when R is rotated about the x-axis. Express the integrand as a function of a single variable.

(c) Set up but do not evaluate an integral representing the volume of the solid generated when R is rotated about the y-axis. Express the integrand as a function of a single variable.

c)



$$V=2\pi\int_0^3 x\left[\sqrt{81-9x^2}-(-3x+9)\right]dx$$

1976 AB3/BC2

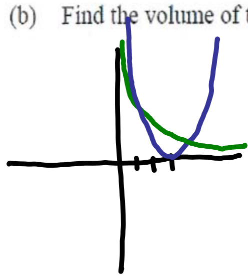
Let R be the region bounded by the curves $f(x)=\frac{4}{x}$ and $g(x)=(x-3)^2$.

$\frac{4}{x}=(x-3)^2$ @ $x=1,\ x=4$

(a) Find the area of R.

(b) Find the volume of the solid generated by revolving R about the x-axis.

a)



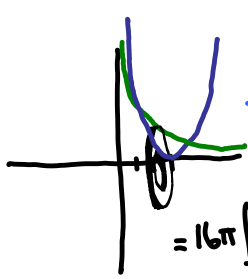
$$A=\int_1^4 \frac{4}{x}-(x-3)^2\,dx$$

$$=\int_1^4 \frac{4}{x}\,dx-\int_1^4 (x-3)^2\,dx \qquad u=x-3 \quad u(4)=1 \qquad du=dx \quad u(1)=-2$$

$$=4\int_1^4 \frac{1}{x}\,dx-\int_{-2}^1 u^2\,du$$

$$=4\left[\ln|x|\right]_1^4-\frac{1}{3}\left[u^3\right]_{-2}^1=4\ln 4-\frac{1}{3}(1-[-8])=\boxed{4\ln 4-3}$$

b)



$$V=\pi\int_1^4 \left(\frac{4}{x}\right)^2-\left[(x-3)^2\right]^2dx=16\pi\int_1^4 x^{-2}\,dx-\pi\int_1^4 (x-3)^4\,dx$$

$u=x-3$, $du=dx$, $u(4)=1$, $u(1)=-2$

$$=16\pi\int_1^4 x^{-2}\,dx-\pi\int_{-2}^1 u^4\,du=16\pi\left[-\frac{1}{x}\right]_1^4-\frac{\pi}{5}\left[u^5\right]_{-2}^1$$

$$=16\pi\left[-\frac{1}{4}-(-1)\right]-\frac{\pi}{5}\left[1-(-32)\right]=12\pi-\frac{33\pi}{5}=\boxed{\frac{27\pi}{5}}$$

1977 AB3

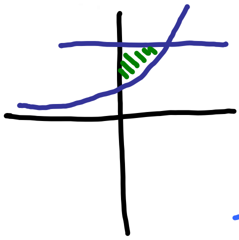
Given the function f defined for all real numbers x by $f(x) = e^{x/2}$.

$e^{x/2} = e^1$

$\frac{x}{2} = 1,\ x = 2$

(a) Find the area of the region R bounded by the line $y = e$, the graph of f, and the y-axis.

(b) Find the volume of the solid generated by revolving R, the region in part (a), about the x-axis.

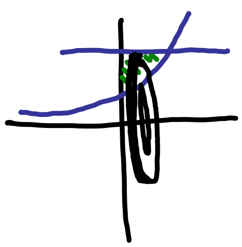


$$A = \int_0^2 e - e^{x/2}\,dx = e\int_0^2 dx - \int_0^2 e^{x/2}\,dx$$

$u = x/2$, $du = \frac{1}{2}dx$, $2du = dx$; $u(2) = 1$, $u(0) = 0$

$$= e\int_0^2 dx - 2\int_0^1 e^u\,du$$

$$= e[x]_0^2 - 2[e^u]_0^1 = 2e - 2[e-1] = 2e - 2e + 2 = \boxed{2}$$



$$V = \pi\int_0^2 e^2 - (e^{x/2})^2\,dx = \pi\int_0^2 e^2 - e^x\,dx$$

$$= \pi\left[e^2x - e^x\right]_0^2 = \pi\left[(2e^2 - e^2) - (0 - 1)\right]$$

$$= \boxed{\pi(e^2 + 1)}$$

1969 AB4/BC4

The number of bacteria in a culture at time t is given approximately by $y = 1000(25 + te^{-t/20})$ for $0 \le t \le 100$. Note the closed interval. The largest and smallest values in (a) refer to abs. max and min respectively.

(a) Find the largest number and the smallest number of bacteria in the culture during the interval. Test the original function at critical values and endpoints.

$$y = 1000(25 + te^{-t/20}) = 25{,}000 + 1000te^{-t/20}$$

$$y' = 1000t \cdot e^{-t/20}\left(-\tfrac{1}{20}\right) + e^{-t/20}(1000)$$

$$= 1000e^{-t/20}\left(-\tfrac{t}{20} + 1\right) = 0 \text{ when } \tfrac{t}{20} = 1 \text{ or } t = 20$$

$y(0) = 25{,}000$ MIN

$y(20) = 1000\left(25 + \frac{20}{e}\right)$ MAX

$y(100) = 1000\left(25 + \frac{100}{e^5}\right)$

(b) At what time during the interval is the rate of change in the number of bacteria a minimum?

$$y' = 1000e^{-t/20}\left(-\tfrac{t}{20} + 1\right)$$

$$y'' = 1000e^{-t/20}\left(-\tfrac{1}{20}\right) + \left(-\tfrac{t}{20} + 1\right)\left(1000e^{-t/20}\right)\left(-\tfrac{1}{20}\right)$$

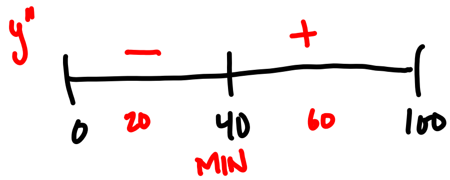
$$= -50e^{-t/20}\left(1 + \left(-\tfrac{t}{20} + 1\right)\right)$$

$$= -50e^{-t/20}\left(2 - \tfrac{t}{20}\right) = 0 \text{ when } 2 = \tfrac{t}{20} \text{ or } t = 40 \text{ sec.}$$

$y'(0) = 1000$

$y'(40) = -\frac{1000}{e^2}$ ← min

$y'(100) = \frac{1000}{e^5}(-4)$



1970 AB1/BC1

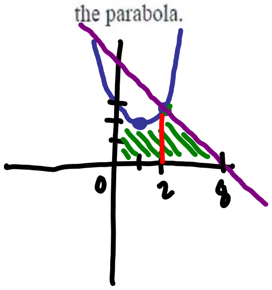
Given the parabola $y = x^2 - 2x + 3$:

$y - y_1 = m(x - x_1)$ $\quad$ $y - 3 = -\frac{1}{2}(x - 2)$

$y = -\frac{1}{2}x + 4$

(a) Find an equation for the line L, which contains the point $(2, 3)$ and is perpendicular to the line tangent to the parabola at $(2, 3)$.

(b) Find the area of that part of the first quadrant which lies below both the line L and the parabola.



$y' = 2x - 2$ @ $(2,3)$ $\quad y' = 2 \quad m = -\frac{1}{2}$

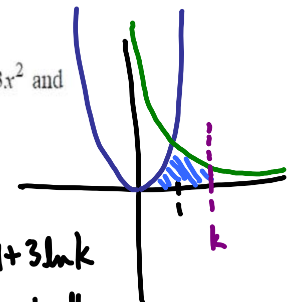
$$A = \int_0^2 x^2 - 2x + 3\,dx + \int_2^8 -\tfrac{1}{2}x + 4\,dx$$

$$= \left[\frac{x^3}{3} - x^2 + 3x\right]_0^2 + \left[-\frac{x^2}{4} + 4x\right]_2^8$$

$$= \left[\left(\frac{8}{3} - 4 + 6\right) - 0\right] + \left[\left(-\frac{64}{4} + 32\right) - (-1 + 8)\right] = \boxed{\frac{41}{3}}$$

1971 AB2

Let R be the region in the first quadrant that lies below both of the curves $y = 3x^2$ and $y = \frac{3}{x}$ and to the left of the line $x = k$ where $k > 1$.



(a) Find the area of R as a function of k.

$$A = \int_0^1 3x^2\,dx + 3\int_1^k \frac{1}{x}\,dx$$

$$= \left[x^3\right]_0^1 + 3\left[\ln|x|\right]_1^k$$

$$= 1 + 3\ln k$$

$1 + 3\ln k = 7$

$3\ln k = 6$

$\ln k = 2$

$e^2 = k$

$A = 1 + 3\ln k$

$\frac{dA}{dt} = 3\frac{1}{k}\frac{dk}{dt}$

$5 = \cancel{\frac{3}{15}}\,\frac{1}{5}\frac{dk}{dt}$

$\frac{dk}{dt} = 25$ units/sec.

(b) When the area of R is 7, what is the value of k?

(c) If the area of R is increasing at the constant rate of 5 square units per second, at what rate is k increasing when $k = 15$?

$\frac{dA}{dt} = 5\ u^2/\text{sec}$ $\quad$ $\frac{dk}{dt} = ?$ when $k = 15$