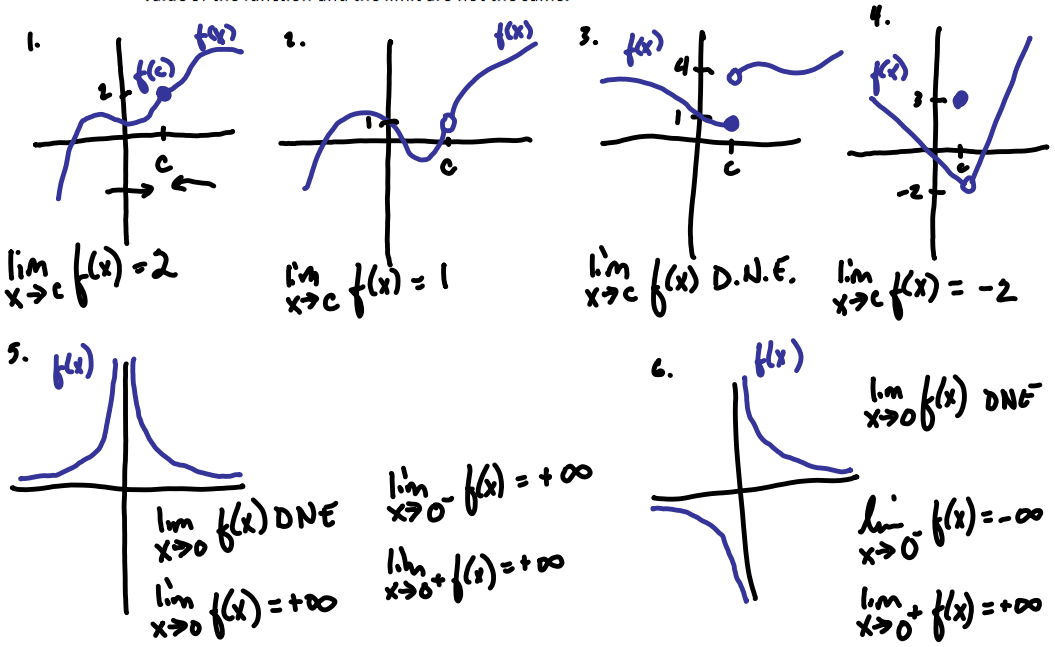
Limits

First type: $\lim_{x \to c} f(x)$

A limit exists as x approaches c if the y-values of $f(x)$ approach the same value from both the left and the right.

Notes: $\lim_{x \to c} f(x) \Rightarrow \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$

1. In most cases, the value of the limit at a certain point and the value of the function are the same. When this happens, the function is continuous at that point.
2. A limit may exist even when a function is not defined for a particular x value.
3. A function may be defined for a particular x value, but the limit does not exist there.
4. A function may be defined for a particular x value, and the limit exists at that point, but the value of the function and the limit are not the same.



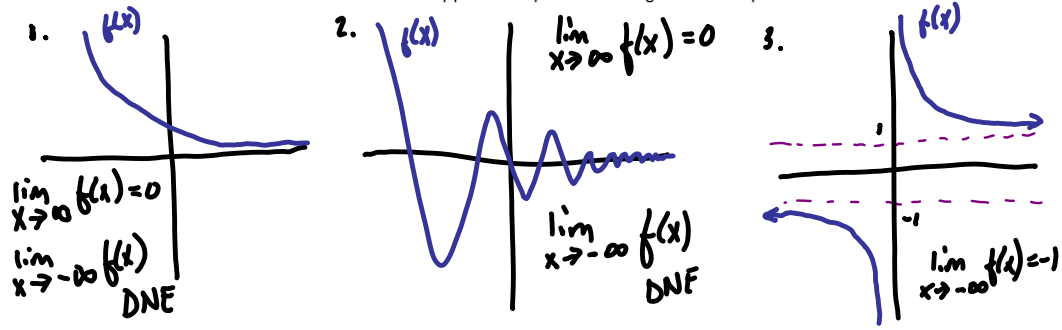
Second type: $\lim_{x \to \infty} f(x)$

A limit exists as x approaches either positive or negative infinity if the y-value of the function gets closer and closer to a particular value and does not grow without bound.

Notes:

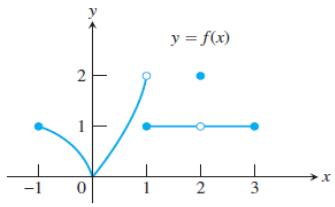
1. When a limit exists as x approaches infinity, we also say that the function has a horizontal asymptote there.
2. A function may never attain the value of the limit as x approaches infinity, but it might.
3. A function can have different limits as it approaches positive or negative infinity.

$\lim_{x \to \infty} f(x) = 1$



Some examples follow. Answers are in light blue.

38.



(a) $\lim_{x\to -1^+} f(x) = 1$ True

(b) $\lim_{x\to 2} f(x)$ does not exist. False

(c) $\lim_{x\to 2} f(x) = 2$ False

(d) $\lim_{x\to 1^-} f(x) = 2$ True

(e) $\lim_{x\to 1^+} f(x) = 1$ True

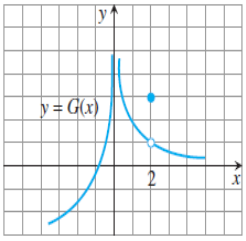
(f) $\lim_{x\to 1} f(x)$ does not exist. True

(g) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$ True

(h) $\lim_{x\to c} f(x)$ exists at every c in $(-1, 1)$. True

(i) $\lim_{x\to c} f(x)$ exists at every c in $(1, 3)$. True

44.



(a) $\lim_{x\to 2^-} G(x)$ 1

(b) $\lim_{x\to 2^+} G(x)$ 1

(c) $\lim_{x\to 2} G(x)$ 1

(d) $G(2)$ 3

Note:
In 44 above, we could have added

$\lim_{x\to 2^-} G(x) = +\infty$

$\lim_{x\to 2^+} G(x) = +\infty$

Finding limits algebraically:

1. Always try direct substitution first (plug-and-chug)
2. Factor and cancel
3. Conjugate method
4. Simplify fractions
5. Determine the behavior of a function around a vertical asymptote
6. Determining limits as x approaches infinity
7. Limits of trigonometric functions

In Exercises 5–22, find the limit.

5. $\lim_{x\to 2} x^4$ 16

6. $\lim_{x\to -2} x^3$ −8

7. $\lim_{x\to 0} (2x - 1)$ −1

8. $\lim_{x\to -3} (3x + 2)$ −7

9. $\lim_{x\to -3} (x^2 + 3x)$ 0

10. $\lim_{x\to 1} (-x^2 + 1)$ 0

11. $\lim_{x\to -3} (2x^2 + 4x + 1)$ 7

12. $\lim_{x\to 1} (3x^3 - 2x^2 + 4)$ 5

13. $\lim_{x\to 2} \frac{1}{x}$ $\frac{1}{2}$

14. $\lim_{x\to -3} \frac{2}{x + 2}$ −2

15. $\lim_{x\to 1} \frac{x - 3}{x^2 + 4}$ $-\frac{2}{5}$

16. $\lim_{x\to 3} \frac{2x - 3}{x + 5}$ 3/8

17. $\lim_{x\to 7} \frac{5x}{\sqrt{x + 2}}$ $\frac{35}{3}$

18. $\lim_{x\to 3} \frac{\sqrt{x + 1}}{x - 4}$ −2

19. $\lim_{x\to 3} \sqrt{x + 1}$ 2

20. $\lim_{x\to 4} \sqrt[3]{x + 4}$ 2

21. $\lim_{x\to -4} (x + 3)^2$ 1

22. $\lim_{x\to 0} (2x - 1)^3$ −1

In Exercises 27–36, find the limit of the trigonometric function.

27. $\lim_{x\to \pi/2} \sin x$ 1

28. $\lim_{x\to \pi} \tan x$ 0

29. $\lim_{x\to 2} \cos \frac{\pi x}{3}$ −1/2

30. $\lim_{x\to 1} \sin \frac{\pi x}{2}$ 1

31. $\lim_{x\to 0} \sec 2x$ 1

32. $\lim_{x\to \pi} \cos 3x$ −1

33. $\lim_{x\to 5\pi/6} \sin x$ 1/2

34. $\lim_{x\to 5\pi/3} \cos x$ 1/2

35. $\lim_{x\to 3} \tan\left(\frac{\pi x}{4}\right)$ −1

36. $\lim_{x\to 7} \sec\left(\frac{\pi x}{6}\right)$ $-\sqrt{3}/2$

In Exercises 49–62, find the limit (if it exists).

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49. $\lim_{x\to 5} \frac{x-5}{x^2-25}$

50. $\lim_{x\to 2} \frac{2-x}{x^2-4}$

$\lim_{x\to 2} \frac{2-x}{x^2-4} = \lim_{x\to 2} \frac{2-x}{(x+2)(x-2)}$

$= \lim_{x\to 2} \frac{1}{x+2} = \boxed{\frac{1}{4}}$

51. $\lim_{x\to -3} \frac{x^2+x-6}{x^2-9}$

52. $\lim_{x\to 4} \frac{x^2-5x+4}{x^2-2x-8}$

53. $\lim_{x\to 0} \frac{\sqrt{x+5}-\sqrt{5}}{x}$

54. $\lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x}$

55. $\lim_{x\to 4} \frac{\sqrt{x+5}-3}{x-4}$

56. $\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3}$

57. $\lim_{x\to 0} \frac{[1/(3+x)]-(1/3)}{x}$

58. $\lim_{x\to 0} \frac{[1/(x+4)]-(1/4)}{x}$

54. $\lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x} \left(\frac{\sqrt{2+x}+\sqrt{2}}{\sqrt{2+x}+\sqrt{2}}\right)$

$= \lim_{x\to 0} \frac{2+x-2}{x(\sqrt{x+2}+\sqrt{2})}$

$= \lim_{x\to 0} \frac{1}{\sqrt{x+2}+\sqrt{2}} = \boxed{\frac{1}{2\sqrt{2}}}$

58. $\lim_{x\to 0} \frac{\frac{1}{x+4}-\frac{1}{4}}{x} \cdot \frac{4(x+4)}{4(x+4)}$

$= \lim_{x\to 0} \frac{4-(x+4)}{4x(x+4)} = \lim_{x\to 0} \frac{-x}{4x(x+4)}$

$= \lim_{x\to 0} \frac{-1}{4(x+4)} = \boxed{-\frac{1}{16}}$

In Exercises 33–48, find the limit.

33. $\lim_{x\to 2^+} \frac{x-3}{x-2}$

34. $\lim_{x\to 1^+} \frac{2+x}{1-x}$

35. $\lim_{x\to 3^+} \frac{x^2}{x^2-9}$

36. $\lim_{x\to 4^-} \frac{x^2}{x^2+16}$

33. $\lim_{x\to 2^+} \frac{x-3}{x-2} = -\infty$ try $x=2.1$ $\frac{-}{+}$

34. $\lim_{x\to 1^+} \frac{2+x}{1-x} = -\infty$ try $x=1.1$ $\frac{+}{-}$

35. $\lim_{x\to 3^+} \frac{x^2}{x^2-9} = +\infty$ try $x=3.1$ $\frac{+}{+}$

36. $\lim_{x\to 4^-} \frac{x^2}{x^2+16} = +\infty$ try $x=3.9$ $\frac{+}{+}$

In Exercises 19–32, find the limit.

19. $\lim_{x\to\infty} \frac{\quad -1}{\quad +2}$ $\frac{4}{3}$

20. $\lim_{x\to\infty} \frac{3x^3+2}{9x^3-2x^2+7}$ $\frac{1}{3}$

21. $\lim_{x\to\infty} \frac{x}{x^2-1}$ 0

22. $\lim_{x\to\infty} \left(4+\frac{3}{x}\right)$ 4

23. $\lim_{x\to -\infty} \frac{5x^2}{x+3}$ $-\infty$

24. $\lim_{x\to -\infty} \left(\frac{1}{2}x-\frac{4}{x^2}\right) = \lim_{x\to -\infty} \frac{x^3-8}{2x^2} = -\infty$

25. $\lim_{x\to -\infty} \frac{x}{\sqrt{x^2-x}}$ -1

26. $\lim_{x\to -\infty} \frac{x}{\sqrt{x^2+1}}$ -1

27. $\lim_{x\to -\infty} \frac{2x+1}{\sqrt{x^2-x}}$ -2

28. $\lim_{x\to -\infty} \frac{-3x+1}{\sqrt{x^2+x}}$ 3

29. $\lim_{x\to\infty} \frac{\sin 2x}{x}$ 0

30. $\lim_{x\to\infty} \frac{x-\cos x}{x} = \lim_{x\to\infty} \frac{x}{x} - \frac{\cos x}{x} = 1$

31. $\lim_{x\to\infty} \frac{1}{2x+\sin x}$ 0

32. $\lim_{x\to\infty} \cos\frac{1}{x} = 1$

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In Exercises 67–78, determine the limit of the trigonometric function (if it exists).

67. $\lim_{x\to 0} \frac{\sin x}{5x}$ $= \frac{1}{5}\lim_{x\to 0}\frac{\sin x}{x} = \frac{1}{5}$

68. $\lim_{x\to 0} \frac{3(1-\cos x)}{x}$ $= 0$

69. $\lim_{x\to 0} \frac{\sin x(1-\cos x)}{2x^2}$

70. $\lim_{\theta\to 0} \frac{\cos\theta \tan\theta}{\theta}$

71. $\lim_{x\to 0} \frac{\sin^2 x}{x}$

72. $\lim_{x\to 0} \frac{\tan^2 x}{x}$

73. $\lim_{h\to 0} \frac{(1-\cos h)^2}{h}$

74. $\lim_{\phi\to\pi} \phi \sec\phi$

75. $\lim_{x\to\pi/2} \frac{\cos x}{\cot x}$

76. $\lim_{x\to\pi/4} \frac{1-\tan x}{\sin x - \cos x}$

77. $\lim_{t\to 0} \frac{\sin 3t}{2t}$

78. $\lim_{x\to 0} \frac{\sin 2x}{\sin 3x}$ $\left[\text{Hint: Find } \lim_{x\to 0}\left(\frac{2\sin 2x}{2x}\right)\left(\frac{3x}{3\sin 3x}\right).\right]$

$\lim_{x\to 0} \frac{\sin x}{x} = 1$

$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

$\lim_{x\to 0} \frac{\tan x}{x} = 1$

69. $\lim_{x\to 0} \frac{1}{2}\frac{\sin x}{x}\frac{(1-\cos x)}{x} = 0$

70. $\lim_{\theta\to 0} \frac{\cos\theta \tan\theta}{1\cdot\theta} = 1$

72. $\lim_{x\to 0} \frac{\tan^2 x}{x} = \lim_{x\to 0} \frac{\tan x}{1}\frac{\tan x}{x} = 0$

74. $\lim_{\varphi\to\pi} \varphi\sec\varphi = \lim_{\varphi\to\pi} \frac{\varphi}{\cos\varphi} = \frac{\pi}{-1} = -\pi$