Answers for Set 3: Differentiation

1. C	**22.** E	**42.** C	**62.** C	**82.** D
2. A	**23.** C	**43.** C	**63.** B	**83.** B
3. B	**24.** A	**44.** A	**64.** B	**84.** C
4. B	**25.** D	**45.** D	**65.** B	**85.** E
5. E	**26.** E	**46.** B	**66.** E	**86.** D
6. B	**27.** D	**47.** E	**67.** B	**87.** C
7. B	**28.** A	**48.** E	**68.** A	**88.** E
8. E	**29.** A	**49.** E	**69.** B	**89.** A
9. D	**30.** D	**50.** A	**70.** C	**90.** E
10. C	**31.** E	**51.** D	**71.** D	**91.** E
11. A	**32.** B	**52.** B	**72.** B	**92.** D
12. B	**33.** C	**53.** C	**73.** D	**93.** B
13. D	**34.** C	**54.** D	**74.** D	**94.** B
14. D	**35.** E	**55.** E	**75.** C	**95.** C
15. A	**36.** B	**56.** D	**76.** B	**96.** C
16. E	**37.** D	**57.** A	**77.** D	**97.** D
17. C	**38.** C	**58.** B	**78.** B	**98.** A
18. E	**39.** A	**59.** E	**79.** B	**99.** D
19. A	**40.** B	**60.** B	**80.** E	**100.** E
20. D	**41.** C	**61.** A	**81.** B	**101.** C
21. D				

Many of the explanations provided include intermediate steps that would normally be reached on the way to a final algebraically simplified result. You may not need to reach the final answer.

NOTE: the formulas or rules in parentheses referred to in the explanations are given on pages 47 and 48.

1. C. By the product rule, (5),

$$\begin{aligned} y' &= (4x+1)[3(1-x)^2(-1)] + (1-x)^3 \cdot 4 \\ &= (1-x)^2(-12x-3+4-4x) \\ &= (1-x)^2(1-16x). \end{aligned}$$

2. A. By the quotient rule, (6),

$$y' = \frac{(3x+1)(-1)-(2-x)(3)}{(3x+1)^2} = -\frac{7}{(3x+1)^2}.$$

3. B. Since $y = (3-2x)^{1/2}$, by the power rule, (3),

$$y' = \frac{1}{2}(3-3x)^{-1/2} \cdot (-2) = -\frac{1}{\sqrt{3-2x}}.$$

4. B. Since $y = 2(5x+1)^{-3}, y' = -6(5x+1)^{-4}(5)$.

5. E. $y' = 3\left(\frac{2}{3}\right)x^{-1/3} - 4\left(\frac{1}{2}\right)x^{-1/2}$

6. B. $(f+2g)'(3) = f'(3) + 2g'(3) = 4 + 2(-1)$

7. B. $(f \cdot g)'(2) = f(2) \cdot g'(2) + g(2) \cdot f'(2) = 5(-2) + 1(3)$

8. E. $\left(\frac{1}{g}\right)'(1) = -1 \cdot \frac{1}{[g(1)]^2} \cdot g'(1) = -1 \cdot \frac{1}{3^2}(-3)$

9. D. $(\sqrt{f})'(3) = \frac{1}{2}[f(3)]^{-1/2} \cdot f'(3) = \frac{1}{2}(10^{-1/2}) \cdot 4$

10. C. $\left(\frac{f}{g}\right)'(0) = \frac{g(0) \cdot f'(0) - f(0) \cdot g'(0)}{[g(0)]^2} = \frac{5(1) - 2(-4)}{5^2}$

11. A. $M'(1) = f'(g(1)) \cdot g'(1) = f'(3)g'(1) = 4(-3)$

12. B. $[f(x^3)]' = f'(x^3) \cdot 3x^2$, so $P'(1) = f'(1^3) \cdot 3 \cdot 1^2 = 2 \cdot 3$.

13. D. $f(S(x)) = x$ implies that $f'(S(x) \cdot S'(x) = 1$, so

$$S'(3) = \frac{1}{f'(S(3))} = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(1)}.$$

14. D. Rewrite: $y = 2x^{1/2} - \frac{1}{2}x^{1/2}$, so $y' = x^{-1/2} + \frac{1}{4}x^{-3/2}$.

15. A. Rewrite: $y = (x^2 + 2x - 1)^{1/2}$. (Use rule (3).)

16. E. Use the quotient rule:

$$y = \frac{\sqrt{1-x^2} \cdot 1 - x \cdot \frac{1 \cdot -2x}{2\sqrt{1-x^2}}}{1-x^2} = \frac{\frac{1-x^2+x^2}{\sqrt{1-x^2}}}{1-x^2}$$

$$= \frac{1}{(1-x^2)^{3/2}}.$$

17. C. Since

$$y = \ln e^x - \ln(e^x - 1)$$
$$= x - \ln(e^x - 1),$$

then

$$y' = 1 - \frac{e^x}{e^x - 1} = \frac{e^x - 1 - e^x}{e^x - 1} = -\frac{1}{e^x - 1}.$$

18. E. Use formula (18): $y' = \frac{\frac{1}{2}}{1 + \frac{x^2}{4}}$.

19. A. Use formulas (13), (11), and (9):

$$y' = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}.$$

20. D. By the quotient rule,

$$y' = \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2}.$$

21. D. Since $y = \ln x + \frac{1}{2}\ln (x^2 + 1)$,

$$y' = \frac{1}{x} + \frac{1}{2} \cdot \frac{2x}{x^2 + 1} = \frac{2x^2 + 1}{x(x^2 + 1)}.$$

22. E. Since $g'(a)$ exists, g is differentiable and thus continuous; $g'(a) > 0$.

23. C. Near a vertical asymptote the slopes must approach $\pm\infty$.

24. A. There is only one horizontal tangent.

25. D. Use the symmetric difference quotient; then

$$f'(1.5) = \frac{f(1.6) - f(1.4)}{1.6 - 1.4} = \frac{8}{0.2}.$$

26. E. Since the water level rises more slowly as the cone fills, the rate of depth change is decreasing, as in (C) and (E). However, at every instant the portion of the cone containing water is similar to the entire cone; the volume is proportional to the cube of the depth of the water. The rate of change of depth (the derivative) is therefore not linear.

27. D. $y' = x^2 \cos\frac{1}{x}\left(-\frac{1}{x^2}\right) + \sin\frac{1}{x}(2x).$

28. A. Since $y = \frac{1}{2}\csc 2x$, $y' = \frac{1}{2}(-\csc 2x \cot 2x \cdot 2)$.

29. A. $y' = e^{-x}(-2\sin 2x) + \cos 2x(-e^{-x}).$

30. D. Use formulas (3) and (11):

$$y' = 2\sec\sqrt{x} \cdot \sec\sqrt{x}\tan\sqrt{x} \cdot \left(\frac{1}{2\sqrt{x}}\right).$$

31. E. $y' = \frac{x(3\ln^2 x)}{x} + \ln^3 x$. The correct answer is $3\ln^2 x + \ln^3 x$.

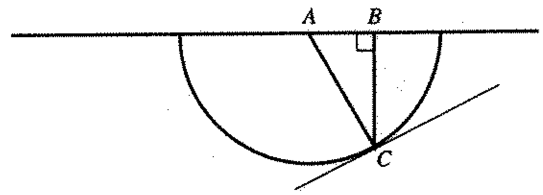
32. B. $y' = \frac{(1-x^2)(2x) - (1+x^2)(-2x)}{(1-x^2)^2}.$

33. C. $y' = \frac{1}{\sqrt{1-x^2}} - \frac{1 \cdot (-2x)}{2\sqrt{1-x^2}}.$

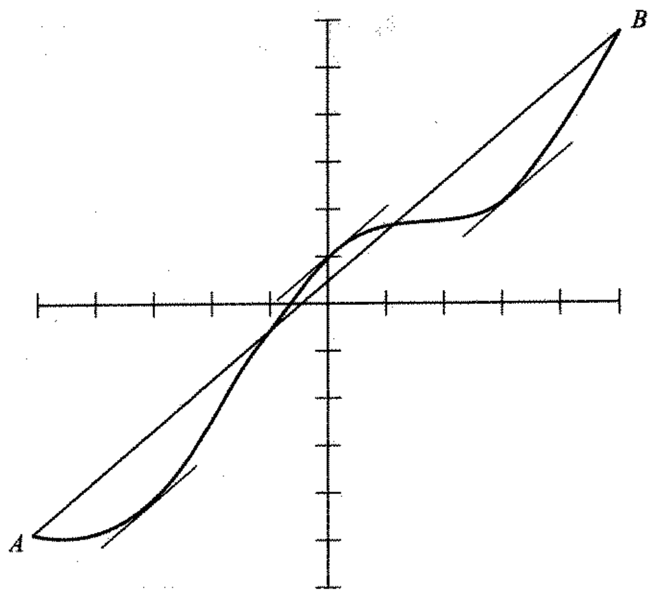
34. C. The only horizontal tangent is at $x = 4$. Note that $f'(1)$ does not exist.

35. E. The graph has corners at $x = 1$ and $x = 2$; the tangent line is vertical at $x = 6$.

36. B. Consider triangle ABC: $AB = 1$; radius $AC = 2$; thus, $BC = \sqrt{3}$ and AC has $m = -\sqrt{3}$. The tangent line is perpendicular to the radius.



37. D. The graph of $y = x + \cos x$ is shown in window $[-5,5] \times [-6,6]$. The average rate of change is represented by the slope of secant segment $\overline{AB}$. There appear to be 3 points at which tangent lines are parallel to $\overline{AB}$.



38. C. $f'(2) \simeq \dfrac{f(2)-f(1.98)}{2-1.98} = \dfrac{4.00-4.10}{0.02}$

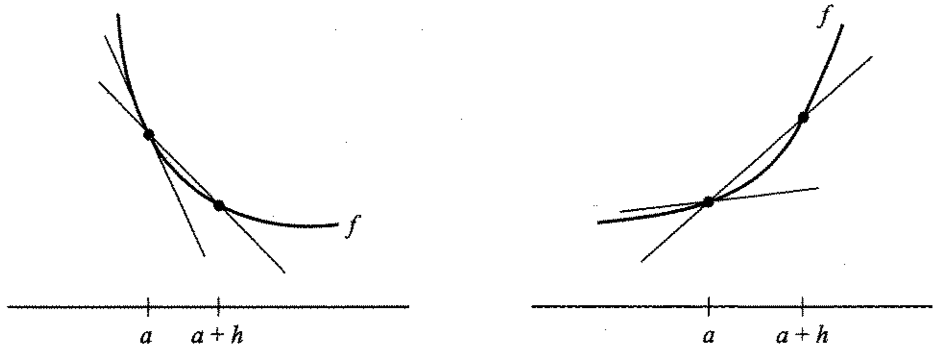
39. A. Since an estimate of the answer for Question 38 is $f'(2) \approx -5$, then $\left(f^{-1}\right)'(4) = \dfrac{1}{f'(2)} \approx \dfrac{1}{-5} = -0.2.$

40. B. When $x = 3$ on g^{-1}, $y = 3$ on the original half-parabola. $3 = x^2 - 8x + 10$ at $x = 1$ (and at $x = 7$, but that value is not in the given domain).

$g'(3) = \dfrac{1}{y'(1)} = \left.\dfrac{1}{2x-8}\right|_{x=1} = -\dfrac{1}{6}.$

41. C. $\dfrac{5^{2.03} - 5^{1.97}}{2.03 - 1.97} \approx 40.25158$

42. C. The diagrams show secant lines (whose slope is the difference quotient) with greater slopes than the tangent line. In both cases, f is concave upward.



43. C. Let y' be $\frac{dy}{dx}$; then $3x^2-(xy'+y)+3y^2y'=0$; $y'(3y^2-x)=y-3x^2$.

44. A. $1-\sin(x+y)(1+y')=0$; $\frac{1-\sin(x+y)}{\sin(x+y)}=y'$.

45. D. $\cos x+\sin y\cdot y'=0$; $y'=-\frac{\cos x}{\sin y}$.

46. B. $6x-2(xy'+y)+10yy'=0$; $y'(10y-2x)=2y-6x$.

47. E. $\frac{dy}{dx}=\frac{4t^3-6t^2}{2t}=2t^2-3t(t\neq 0)$; $\frac{d^2y}{dx^2}=\frac{4t-3}{2t}$. Replace t by 1.

48. E. $f'(x)=4x^3-12x^2+8x=4x(x-1)(x-2)$.

49. E. $f'(x)=8x^{-1/2}$; $f''(x)=-4x^{-3/2}=-\frac{4}{x^{3/2}}$; $f''(4)=-\frac{4}{8}$.

50. A. $f(x)=3\ln x$; $f'(x)=\frac{3}{x}$; $f''(x)=\frac{-3}{x^2}$. Replace x by 3.

51. D. $2x+2yy'=0$; $y'=-\frac{x}{y}$; $y''=-\frac{y-xy'}{y^2}$. At (0,5), $y''=-\frac{5-0}{25}$.

52. B. $y'=ac\cos ct-bc\sin ct$;
$y''=-ac^2\sin ct-bc^2\cos ct$.

53. C. $(f\circ g)'$ at $x=3$ equals $f'(g(3))\cdot g'(3)$ equals $\cos u$ (at $u=0$) times $2x$ (at $x=3$) $=1\cdot 6=6$.

54. D. $f'(1)\approx\frac{5^{1.002}-5^1}{0.002}=\frac{5.016-5}{0.002}$.

55. E. Here $f'(x)$ equals $\frac{-x^2-x-1}{(x-1)^3}$.

56. D. $y'=e^x\cdot 1+e^x(x-1)=xe^x$
$y''=xe^x+e^x$ and $y''(0)=0\cdot 1+1=1$.

57. A. $\dfrac{dy^2}{dx^2} = \dfrac{\frac{dy^2}{dx}}{\frac{dx^2}{dx}}$. Since $y^2 = x^2 + 1$, $\dfrac{dy^2}{dx^2} = \dfrac{2x}{2x}$.

58. B. Note that $f(g(x)) = \dfrac{1}{x+1}$.

59. E. When simplified, $\dfrac{dy}{dx} = \dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$.

60. B. Since (if $\sin t \neq 0$)

$$\frac{dy}{dt} = -2\sin 2t = -4\sin t\cos t \text{ and } \frac{dx}{dt} = -\sin t,$$

then $\dfrac{dy}{dx} = 4\cos t$. Thus:

$$\frac{d^2y}{dx^2} = -\frac{4\sin t}{-\sin t}.$$

61. A. $\dfrac{dy}{d\left(\frac{1}{1-x}\right)} = \dfrac{\frac{dy}{dx}}{\frac{d\left(\frac{1}{1-x}\right)}{dx}} = \dfrac{2x+1}{\frac{1}{(1-x)^2}}$.

NOTE: Since each of the limits in Questions 62 through 65 yields an indeterminate form of the type $\frac{0}{0}$, we can apply L'Hôpital's rule in each case, getting identical answers

62. C. The given limit is the derivative of $f(x) = x^6$ at $x = 1$.

63. B. The given limit is the definition for $f'(8)$, where $f(x) = \sqrt[3]{x}$;

$$f'(x) = \frac{1}{3x^{2/3}}.$$

64. B. The given limit is $f'(e)$, where $f(x) = \ln x$.

65. B. The given limit is the derivative of $f(x) = \cos x$ at $x = 0$; $f'(x) = -\sin x$.

66. E. Since $f'(x) = \dfrac{2}{3x^{1/3}}$, $f'(0)$ is not defined; $f'(x)$ must be defined on $(-8,8)$.

67. B. Sketch the graph of $f(x) = 1 - |x|$; note that $f(-1) = f(1) = 0$ and that f is continuous on $[-1,1]$. Only (B) holds.

68. A. Note that $f(0) = f(\sqrt{3}) = 0$ and that $f'(x)$ exists on the given interval. By the MVT, there is a number c in the interval such that $f'(c) = 0$. If $c = 1$, then $6c^2 - 6 = 0$. (-1 is not in the interval.)

69. B. Since the inverse, h, of $f(x) = \dfrac{1}{x}$ is $h(x) = \dfrac{1}{x}$, then $h'(x) = -\dfrac{1}{x^2}$. Replace by 3.

70. C. Since $f'(x) = 6x^2 - 3$, therefore $h'(x) = \frac{1}{6x^2-3}$; also, $f(x)$, or $2x^3 - 3x$, equals -1, by observation, for $x = 1$. So $h'(-1)$ or $\frac{1}{6x^2-3}$ (when $x = 1$) equals $\frac{1}{6-3} = \frac{1}{3}$.

71. D. $\left(f^{-1}\right)'(2) = \frac{1}{f'(1)} = \frac{1}{3}$.

72. B. Since $f(0) = 5$, $g'(5) = \frac{1}{f'(0)} = \left.\frac{1}{3x^2 - 6x + 8}\right|_{x=0} = \frac{1}{8}$.

73. D. After 50(!) applications of L'Hôpital's rule we get $\lim\limits_{x\to\infty} \frac{e^x}{50!}$, which "equals" ∞. A perfunctory examination of the limit, however, shows immediately that the answer is ∞. In fact, $\lim\limits_{x\to\infty} \frac{e^x}{x^n}$ for any positive integer n, no matter how large, is ∞.

74. D. The given limit is the derivative of $g(x)$ at $x = 0$.

75. C. $\cos(xy)(xy' + y) = 1;\ x\cos(xy)y' = 1 - y\cos(xy);$

$$y' = \frac{1 - y\cos(xy)}{x\cos(xy)}.$$

76. B. The tangent line appears to contain (3,–2.6) and (4,–1.8).

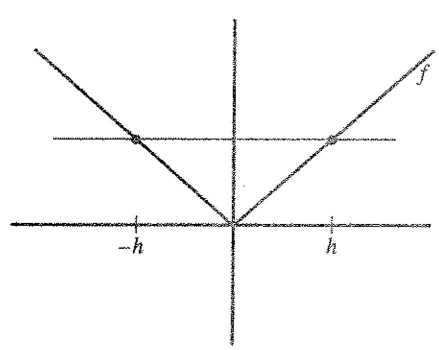
77. D. $f'(x)$ is least at the point of inflection of the curve, at about 0.7.

78. B. Note that $\frac{(x+h)^2 - (x-h)^2}{2h} = 2x$, the derivative of $f(x) = x^2$. You should be able to confirm that the symmetric difference quotient works for the general quadratic $g(x) = ax^2 + bx + c$.

79. B. By calculator, $f'(0) = 1.386294805$ and $\frac{4^{0.08} - 4^{-0.08}}{0.16} = 1.3891\ldots$.

80. E. Now $\frac{4^{0.001} - 4^{-0.001}}{0.002} = 1.386294805$.

81. B. Note that any line determined by two points equidistant from the origin will necessarily be horizontal.



82. D. Note that $\frac{d}{dx} f(h(x)) = f'(h(x)) \cdot h'(x) = g(h(x)) \cdot h'(x) = g(\sin x) \cdot \cos x$.

NOTE: In Questions 83 through 87 the limits are all indeterminate forms of the type $\frac{0}{0}$. We have therefore applied L'Hôpital's rule in each one. The indeterminacy can also be resolved by introducing $\frac{\sin a}{a}$, which approaches 1 as a approaches 0. The latter technique is presented in square brackets.

83. B. $\lim_{x\to0} \frac{\sin 2x}{x} = \lim_{x\to0} \frac{2\cos 2x}{1} = \frac{2 \cdot 1}{1} = 2.$

[Using $\sin 2x = 2 \sin x \cos x$ yields $\lim_{x\to0} 2\left(\frac{\sin x}{x}\right)\cos x = 2 \cdot 1 \cdot 1 = 2.$]

84. C. $\lim_{x\to0} \frac{\sin 3x}{\sin 4x} = \lim_{x\to0} \frac{3\cos 3x}{4\cos 4x} = \frac{3 \cdot 1}{4 \cdot 1} = \frac{3}{4}$.

[We rewrite $\frac{\sin 3x}{\sin 4x}$ as $\frac{\sin 3x}{3x} \cdot \frac{4x}{\sin 4x} \cdot \frac{3}{4}$. As $x \to 0$, so do $3x$ and $4x$; the fraction approaches $1 \cdot 1 \cdot \frac{3}{4}$.]

85. E. $\lim_{x\to0} \frac{1-\cos x}{x} = \lim_{x\to0} \frac{\sin x}{1} = 0.$

[We can replace $1 - \cos x$ by $2\sin^2 \frac{x}{2}$, getting

$$\lim_{x\to0} \frac{2\sin^2 \frac{x}{2}}{x} = \lim_{x\to0} \frac{\sin^2 \frac{x}{2}}{\frac{x}{2}} = \lim_{x\to0} \sin\frac{x}{2} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right) = 0 \cdot 1.]$$

86. D. $\lim_{x\to0} \frac{\tan \pi x}{x} = \lim_{x\to0} \frac{(\sec^2 \pi x) \cdot \pi}{1} = 1 \cdot \pi = \pi.$

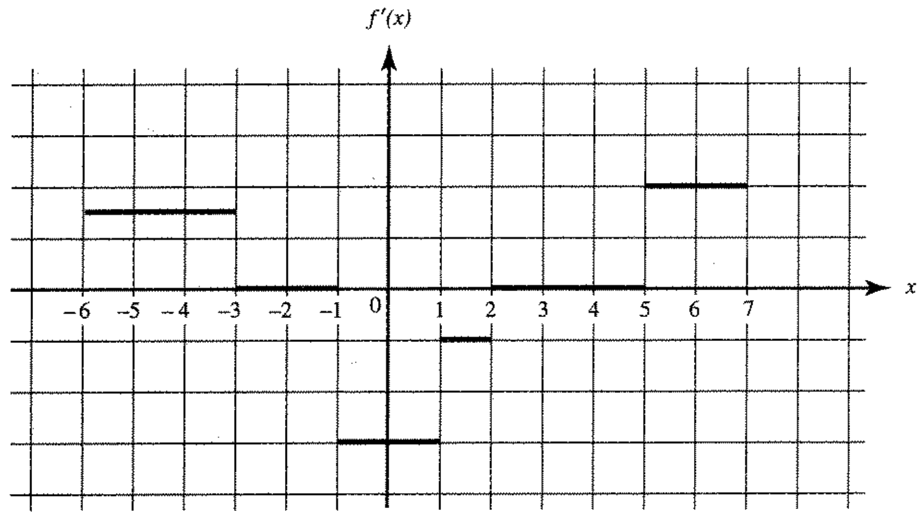
[$\frac{\tan \pi x}{x} = \frac{\sin \pi x}{x \cos \pi x} = \pi \cdot \frac{\sin \pi x}{\pi x} \cdot \frac{1}{\cos x}$; as x (or πx) approaches 0, the original fraction approaches $\pi \cdot 1 \cdot \frac{1}{1} = \pi$.]

87. C. The limit is easiest to obtain here if we rewrite:

$$\lim_{x\to\infty} x^2 \sin\frac{1}{x} = \lim_{x\to\infty} x \frac{\sin(1/x)}{(1/x)} = \infty \cdot 1 = \infty.$$

88. E. Since $f(x) = 3^x - x^3$, then $f'(x) = 3^x \ln 3 - 3x^2$. Furthermore, f is continuous on [0,3] and f' is differentiable on (0,3), so the MVT applies. We therefore seek c such that $f'(c) = \frac{f(3)-f(0)}{3} = -\frac{1}{3}$. Using a graphing calculator, we key in $Y_1 = 3^X \ln 3 - 3X^2 + \frac{1}{3}$. With solutions estimated at $x = 1$ and 2.5, th calculator shows that c may be either 1.244 or 2.727. These are the x-coordinates of points on the graph of $f(x)$ at which the tangents are parallel to the secant through points (0,1) and (3,0) on the curve.

89. A. The line segment passes through (1,–3) and (2,–4).



Use the graph of $f'(x)$, shown above, for Questions 90 through 93.

90. E. $f'(x) = 0$ when the slope of $f(x)$ is 0; that is, when the graph of f is a horizontal segment.

91. E. The graph of $f'(x)$ jumps at each corner of the graph of $f(x)$, namely, at x equal to –3, –1, 1, 2, and 5.

92. D. On the interval (–6,–3), $f(x) = \frac{3}{2}(x + 5)$.

93. B. Verify that all choices but (B) are true. The graph of $f'(x)$ has five (not four) jump discontinuities.

94. B. Since $x - 3 = 2 \sin t$ and $y + 1 = 2 \cos t$, we have

$$(x - 3)^2 + (y + 1)^2 = 4.$$

This is the equation of a circle with center at (3,–1) and radius 2. In the domain given, $-\pi \leqslant t \leqslant \pi$, the entire circle is traced by a particle moving counterclockwise, starting from and returning to (3, –3).

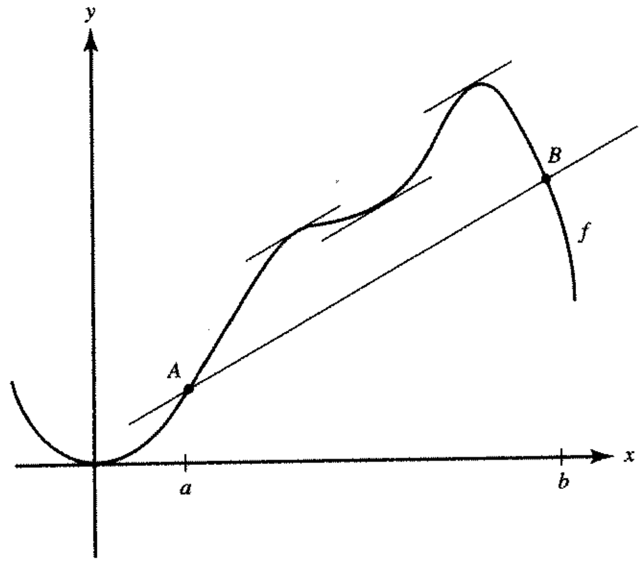
95. C. Use L'Hôpital's rule; then

$$\lim_{x\to 0}\frac{\sec x - \cos x}{x^2} = \lim_{x\to 0}\frac{\sec x \tan x + \sin x}{2x}$$

$$= \lim_{x\to 0}\frac{\sec^3 x + \sec x \tan^2 x + \cos x}{2} = \frac{1+1\cdot 0+1}{2} = 1.$$

96. C. The best approximation to $f'(0.10)$ is $\frac{f(0.20)-f(0.10)}{0.20-0.10}$.

97. D.



The average rate of change is represented by the slope of secant segment $\overline{AB}$. There appear to be 3 points at which the tangent lines are parallel to $\overline{AB}$.

98. A. $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\sin t}{1-\cos t}$.

99. D. $\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{3\sin^2\theta\cos\theta}{-3\cos^2\theta\sin\theta}$.

100. E. $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{1-e^{-t}}{e^{-t}}=e^t-1$.

101. C. Since $\frac{dy}{dt}=\frac{1}{1-t}$ and $\frac{dx}{dt}=\frac{1}{(1-t)^2}$, then

$$\frac{dy}{dx}=1-t=\frac{1}{x}.$$