Section 5.4

## Problem Set 5.4

**1.** $f(x) = 4x^{3/2}, f'(x) = 6x^{1/2}$

$$L = \int_{1/3}^{5} \sqrt{1+(6x^{1/2})^2}\, dx = \int_{1/3}^{5} \sqrt{1+36x}\, dx$$

$$= \left[\frac{1}{36} \cdot \frac{2}{3}(1+36x)^{3/2}\right]_{1/3}^{5}$$

$$= \frac{1}{54}\left(181\sqrt{181} - 13\sqrt{13}\right) \approx 44.23$$

**3.** $f(x) = (4 - x^{2/3})^{3/2}$,

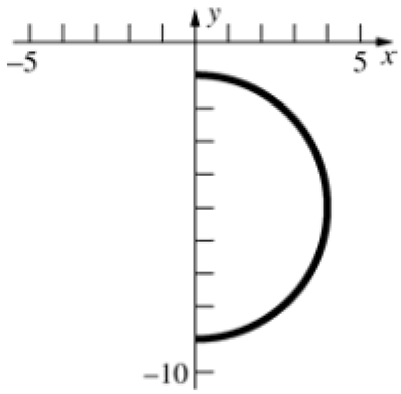
$$f'(x) = \frac{3}{2}(4 - x^{2/3})^{1/2}\left(-\frac{2}{3}x^{-1/3}\right)$$

$$= -x^{-1/3}(4 - x^{2/3})^{1/2}$$

$$L = \int_{1}^{8} \sqrt{1 + \left[-x^{-1/3}(4 - x^{2/3})^{1/2}\right]^2}\, dx$$

$$= \int_{1}^{8} \sqrt{4x^{-2/3}}\, dx = \int_{1}^{8} 2x^{-1/3} dx$$

$$= 2\left[\frac{3}{2}x^{2/3}\right]_{1}^{8} = 3(4-1) = 9$$

**9.** 


$$\frac{dx}{dt} = 4\cos t, \frac{dy}{dt} = -4\sin t$$

$$L = \int_0^{\pi} \sqrt{(4\cos t)^2 + (-4\sin t)^2}\, dt$$

$$= \int_0^{\pi} \sqrt{16\cos^2 t + 16\sin^2 t}\, dt = \int_0^{\pi} 4dt$$

$$= 4\pi \approx 12.57$$

**11.** $f(x) = 2x + 3, f'(x) = 2$

$$L = \int_1^3 \sqrt{1+(2)^2}\, dx = \sqrt{5}\int_1^3 dx = 2\sqrt{5}$$

At $x = 1, y = 2(1) + 3 = 5$.

At $x = 3, y = 2(3) + 3 = 9$.

$$d = \sqrt{(3-1)^2 + (9-5)^2} = \sqrt{20} = 2\sqrt{5}$$

**18.** **a.** $\overline{OT} = \text{length}\left(\widehat{PT}\right) = a\theta$

**b.** From Figure 18 of the text,

$$\sin\theta = \frac{\overline{PQ}}{\overline{PC}} = \frac{\overline{PQ}}{a} \text{ and } \cos\theta = \frac{\overline{QC}}{\overline{PC}} = \frac{\overline{QC}}{a}.$$

Therefore $\overline{PQ} = a\sin\theta$ and $\overline{QC} = a\cos\theta$.

**c.** $x = \overline{OT} - \overline{PQ} = a\theta - a\sin\theta = a(\theta - \sin\theta)$

$y = \overline{CT} - \overline{CQ} = a - a\cos\theta = a(1 - \cos\theta)$

**19.** From Problem 18,

$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$

$\frac{dx}{d\theta} = a(1 - \cos\theta), \frac{dy}{d\theta} = a\sin\theta$ so

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = [a(1 - \cos\theta)]^2 + [a\sin\theta]^2$$

$$= a^2 - 2a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta$$

$$= 2a^2 - 2a^2\cos\theta = 2a^2(1 - \cos\theta)$$

$$= 4a^2\frac{1 - \cos\theta}{2} = 4a^2\sin^2\left(\frac{\theta}{2}\right).$$

The length of one arch of the cycloid is

$$\int_0^{2\pi}\sqrt{4a^2\sin^2\left(\frac{\theta}{2}\right)}\,d\theta = \int_0^{2\pi} 2a\sin\left(\frac{\theta}{2}\right)d\theta$$

$$= 2a\left[-2\cos\frac{\theta}{2}\right]_0^{2\pi} = 2a(2 + 2) = 8a$$

**21. a.** $\frac{dy}{dx} = \sqrt{x^3 - 1}$

$L = \int_1^2 \sqrt{1 + x^3 - 1}\, dx = \int_1^2 x^{3/2} dx$

$= \left[ \frac{2}{5} x^{5/2} \right]_1^2 = \frac{2}{5}\left(4\sqrt{2} - 1\right) \approx 1.86$

**b.** $f'(t) = 1 - \cos t,\ g'(t) = \sin t$

$L = \int_0^{4\pi} \sqrt{2 - 2\cos t}\, dt = \int_0^{4\pi} 2\left|\sin\left(\frac{t}{2}\right)\right| dt$

$\sin\left(\frac{t}{2}\right)$ is positive for $0 < t < 2\pi$, and by symmetry, we can double the integral from 0 to $2\pi$.

$L = 4\int_0^{2\pi} \sin\left(\frac{t}{2}\right) dt = \left[ -8\cos\frac{t}{2} \right]_0^{2\pi}$

$= 8 + 8 = 16$

Section 5.5

## Problem Set 5.5

**1.** $F\left(\frac{1}{2}\right) = 6;\ k \cdot \frac{1}{2} = 6,\ k = 12$

$F(x) = 12x$

$W = \int_0^{1/2} 12x\, dx = \left[ 6x^2 \right]_0^{1/2} = \frac{3}{2} = 1.5$ ft-lb

**2.** From Problem 1, $F(x) = 12x$.

$W = \int_0^2 12x\, dx = \left[ 6x^2 \right]_0^2 = 24$ ft-lb

**3.** $F(0.01) = 0.6;\ k = 60$

$F(x) = 60x$

$W = \int_0^{0.02} 60x\, dx = \left[ 30x^2 \right]_0^{0.02} = 0.012$ Joules

**9.** A slab of thickness $\Delta y$ at height $y$ has width $4-\frac{4}{5}y$ and length 10. The slab will be lifted a distance $10-y$.

$$\Delta W \approx \delta \cdot 10 \cdot \left(4-\frac{4}{5}y\right)\Delta y(10-y)$$

$$= 8\delta(y^2-15y+50)\Delta y$$

$$W = \int_0^5 8\delta(y^2-15y+50)\,dy$$

$$= 8(62.4)\left[\frac{1}{3}y^3-\frac{15}{2}y^2+50y\right]_0^5$$

$$= 8(62.4)\left(\frac{125}{3}-\frac{375}{2}+250\right) = 52{,}000 \text{ ft-lb}$$

**11.** A slab of thickness $\Delta y$ at height $y$ has width $\frac{3}{4}y+3$ and length 10. The slab will be lifted a distance $9-y$. $\Delta W \approx \delta \cdot 10 \cdot \left(\frac{3}{4}y+3\right)\Delta y(9-y)$

$$= \frac{15}{2}\delta(36+5y-y^2)\Delta y$$

$$W = \int_0^4 \frac{15}{2}\delta(36+5y-y^2)dy$$

$$= \frac{15}{2}(62.4)\left[36y+\frac{5}{2}y^2-\frac{1}{3}y^3\right]_0^4$$

$$= \frac{15}{2}(62.4)\left(144+40-\frac{64}{3}\right)$$

$$= 76{,}128 \text{ ft-lb}$$

**19.** The total work is equal to the work $W_1$ to haul the load by itself and the work $W_2$ to haul the rope by itself.

$W_1 = 200 \cdot 500 = 100,000$ ft-lb

Let $y = 0$ be the bottom of the shaft. When the rope is at $y$, $\Delta W_2 \approx 2\Delta y(500 - y)$.

$$W_2 = \int_0^{500} 2(500 - y)dy = 2\left[500y - \frac{1}{2}y^2\right]_0^{500}$$

$= 2(250,000 - 125,000) = 250,000$ ft-lb

$W = W_1 + W_2 = 100,000 + 250,000$

$= 350,000$ ft-lb

**21.** $f(x) = \frac{k}{x^2}$; $f(4000) = 5000$

$\frac{k}{4000^2} = 5000$, $k = 80,000,000,000$

$$W = \int_{4000}^{4200} \frac{80,000,000,000}{x^2} dx$$

$$= 80,000,000,000\left[-\frac{1}{x}\right]_{4000}^{4200}$$

$$= \frac{20,000,000}{21} \approx 952,381 \text{ mi-lb}$$